Parametric equations.

$\odot = \begin{cases} x = a\cos\phi \\ y = b\sin\phi \end{cases}$

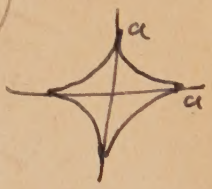
$\supset C = \begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$ ←

$\overset{\cup}{\cap} = \begin{cases} x = b\tan\theta \\ y = a\sec\theta \end{cases}$ } conjugate of

cycloid $y = a(1 - \cos\phi)$

$x = a(\phi - \sin\phi)$

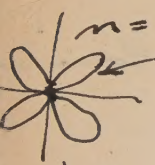
$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$



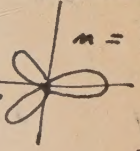
Walter R. Fischer
2139 Lakeland, Ave
#1024 Madison, Wis
Ralph Schmiedlin
2810 Oakridge Ave
Madison, Wis.

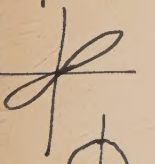
sin

cos.

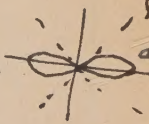


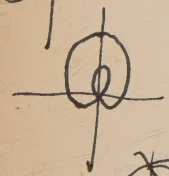
$\rho = a \begin{pmatrix} \sin \\ \cos \end{pmatrix} n\theta$



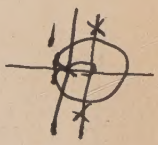


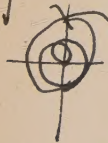
$\rho^2 = a^2 \begin{pmatrix} \sin \\ \cos \end{pmatrix} 2\theta$





$\rho = a \begin{pmatrix} \sin \\ \cos \end{pmatrix} \theta + b$

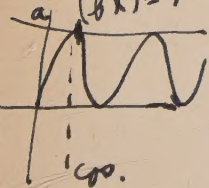




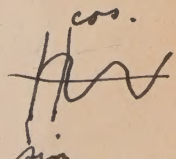
$\rho = \theta \quad \rho = \theta^2$

spirals

cos.

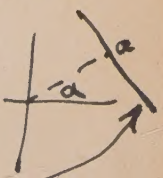


$y = a \begin{matrix} \sin \\ \cos \end{matrix} (\beta x - K)$



$y = a \begin{pmatrix} \sin^2 \\ \cos^2 \end{pmatrix} \beta x \longleftarrow x = 0$

$x \cos\alpha + y \sin\alpha = a$



$a = \rho \cos\theta$

solutions of △ by log.

Case	given	Logs	No logs.
I	2 ∠, side	Laws of Sines	Laws of Sines
II	2 sides, ∠ not incl	" " "	" " "
III	2 sides, incl ∠	" of Tangents	Cosines
IV	3 sides	S-formulas	Cosines
		don't use Cosines	

ELEMENTARY MATHEMATICAL ANALYSIS

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ELEMENTARY MATHEMATICAL ANALYSIS

A TEXT BOOK FOR FIRST YEAR COLLEGE STUDENTS

BY

CHARLES S. SLICHTER, Sc. D.

Professor of Applied Mathematics, University of Wisconsin

Entirely Reset With Revisions and Additions by

WARREN WEAVER, PH. D.

Associate Professor of Mathematics, University of Wisconsin

THIRD EDITION
FOURTH IMPRESSION

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PREFACE TO THE THIRD EDITION

This edition of *Elementary Mathematical Analysis* has been revised by Dr. Warren Weaver, Associate Professor of Mathematics in the University of Wisconsin. The rearrangement of material and all other changes have, in practically every case, been discussed with those who have been using the text in the class room. The effort has been to secure independence of judgment at all points in the revision and Professor Weaver has carried through the work, in the opinion of the writer, in a genuinely constructive manner. The changes made in the chapter on the Circular Functions and in the chapter on the Logarithmic Function are especially notable.

Exercises have been increased in number and rearranged so that such work may be assigned day by day with greater continuity. There are exercises in sufficient variety to enable the instructor to keep the very bright, as well as the average student, fully occupied with the work.

The text has been shortened by about thirty pages, but several hundred exercises have been added to the review chapter on Elementary Algebra.

CHARLES S. SLICHTER.

MADISON, WIS.
July, 1925.

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FROM THE PREFACE TO THE FIRST EDITION

THIS book is not intended to be a text on "Practical Mathematics" in the sense of making use of scientific material and of fundamental notions not already in the possession of the student, or in the sense of making the principles of mathematics secondary to its technique. On the contrary, it has been the aim to give the fundamental truths of elementary analysis as much prominence as seems possible in a working course for freshmen.

The emphasis of the book is intended to be upon the notion of *functionality*. Illustrations from science are freely used to make this concept prominent. The student should learn early in his course that an important purpose of mathematics is to express and to interpret the laws of actual phenomena and not primarily to secure here and there certain computed results. Mathematics might well be defined as the science that takes the broadest view of all of the sciences—an epitome of quantitative knowledge. The introduction of the student to a broad view of mathematics can hardly begin too early.

The ideas explained above are developed in accordance with a two-fold plan, as follows:

First, the plan is to group the material of elementary analysis about the consideration of the three fundamental functions:

1. The Power Function $y = ax^n$ (n any number) or the law "*as x changes by a fixed multiple, y changes by a fixed multiple also.*"

2. The Simple Periodic Function $y = a \sin mx$, considered as fundamental to all periodic phenomena.

3. The Exponential Function, or the law "*as x changes by a fixed increment, y changes by a fixed multiple.*"

Second, the plan is to enlarge the elementary functions by the development of the fundamental *transformations* applicable to these and other functions. To avoid the appearance of abstruse-

ness, these transformations are stated with respect to the *graphs* of the functions; that is, they are not called *transformations*, but "motions" of the loci. The facts are summarized in several simple "Theorems on Loci," which explain the translation, rotation, shear, and elongation or contraction of the graph of any function in the xy plane.

Combinations of the fundamental functions as they actually occur in the expression of elementary natural laws are also discussed and examples are given of a type that should help to explain their usefulness.

Emphasis is placed upon the use of *time* as variable. This enriches the treatment of the elementary functions and brings many of the facts of "analytic geometry" into close relation to their application in science. A chapter on waves is intended to give the student a broad view of the use of the trigonometric functions and an introduction to the application of analysis to periodic phenomena.

It is difficult to understand why it is customary to introduce the trigonometric functions to students seventeen or eighteen years of age by means of the restricted definitions applicable only to the right triangle. Actual test shows that such rudimentary methods are wasteful of time and actually confirm the student in narrowness of view and in lack of scientific imagination. For that reason, the definitions, theorems and addition formulas of trigonometry are kept as general as practicable and the formulas are given general demonstrations.

The possibilities and responsibilities of character building in the department of mathematics are kept constantly in mind. It is accepted as fundamental that a modern working course in mathematics should emphasize proper habits of work as well as proper methods of thought; that neatness, system, and orderly habits have a high value to all students of the sciences, and that a textbook should help the teacher in every known way to develop these in the student.

The present work is a revision and rewriting of a preliminary form which has been in use for three years at the University of Wisconsin. During this time the writer has had frequent and valuable assistance from the instructional force of the department

of mathematics in the revision and betterment of the text. Acknowledgments are due especially to Professors Burgess, Dresden, Hart and Wolff and to Instructors Fry, Nyberg and Taylor. Professor Burgess has tested the text in correspondence courses, and has kindly embraced that opportunity to aid very materially in the revision. He has been especially successful in shortening graphical methods and in adapting them to work on squared paper. Professor Wolff has read all of the final manuscript and made many suggestions based upon the use of the text in the class room. Mr. Taylor has read all of the proof and supplied the results to the exercises.

Professor E. V. Huntington of Harvard University has read the galley proof and has contributed many important suggestions.

The writer has avoided the introduction of new technical terms, or terms used in an unusual sense. He has taken the liberty, however of naming the function ax^n, the "Power Function of x," as a short name for this important function seems to be an unfortunate lack—a lack, which is apparently confined solely to the English language.

Charles S. Slichter.

University of Wisconsin
July 25, 1914

Note: The results to the exercises are issued as a separate pamphlet.

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INTRODUCTION

Any course in mathematics requires the frequent use of geometrical constructions, and the carrying out of analytical and numerical computations. In order that this work may be performed neatly and accurately it is necessary that the student have a few simple instruments, and a supply of proper material for doing the work in a systematic and orderly manner. The indispensible instruments are as follows:

I. Instruments. (1) Two $4H$ hexagonal drawing pencils; one sharpened to a fine point for marking points upon paper or for sketching free hand; the other sharpened to a chisel point for drawing straight lines. Some prefer to use a single pencil sharpened at both ends, one end round pointed, the other end chisel pointed.

(2) A small drawing board[1] of soft wood—10×12 inches is large enough.

(3) A small T-square same length as the drawing board.

(4) A 60° and a 45° transparent triangle. Five-inch triangles are large enough, although a larger 60° triangle will be found to be very convenient.

(5) A protractor for laying off angles.

(6) A triangular boxwood scale, decimally divided.

(7) A pair of 6-inch pencil compasses for drawing circles and arcs of circles, provided with medium hard lead, sharpened to a narrow chisel point.

(8) A 10-inch slide rule is required for Chapter IX, and may be used earlier at the discretion of the instructor.

II. Materials. All mathematical work should be done on one side of standard size letter paper, $8\frac{1}{2} \times 11$ inches. This is the smallest sheet that permits proper arrangement of mathematical work. A good equipment will include:

(1) A note book cover to hold sheets of the above named size and

[1]Drawing boards of this size with T-square and two wood triangles are marketed by the Milton Bradly Co., Springfield, Mass.

a supply of manila paper "vertical file folders" for use in submitting work for the examination of the instructor.

(2) A number of different forms of squared paper and computation paper especially prepared for use with this book. These sheets will be described from time to time as needed in the work. Form *M*2 will be found convenient for problem work and for general calculation. *M*2 is a copy of a form used by a number of public utility and industrial corporations. Colleges usually have their own sources of supply of squared paper, satisfactory for use with this book.

(3) Miscellaneous supplies such as thumb tacks, erasers, sandpaper-pencil-sharpeners, etc.

III. General Directions. All drawings should be done in pencil, unless the student has had training in the use of the ruling pen, in which case he may, if he desires, "ink in" a few of the most important drawings.

All mathematical work, such as the solutions of problems and exercises, and work in computation *should be done in ink.* The student should acquire the habit of working problems with pen and ink. He will find that this habit will materially aid him in repressing carelessness and indifference and in acquiring neatness and system.

TO THE STUDENT—SUGGESTIONS ON THE STUDY OF MATHEMATICS

The following suggestions may assist the student to acquire habits of work essential to success in the study of mathematics and of the other exact sciences.

Successful intellectual work depends very largely upon *the power of concentration.* Fortunately this power can be acquired and cultivated. The student should study away from interruption and then must not permit his work to become interrupted by himself or by others. By holding his attention upon his work and by keeping his mind from wandering to extraneous matters, the student will cultivate a fundamental habit that will tend to assure his success both in and out of college.

In a course in mathematics a student (1) *studies a textbook* and (2) *works exercises and problems.* An assignment for a given day may therefore consist of the study of mathematical principles and theory (such as theorems, definitions, and explanations of processes), or it may consist of the working out of exercises and problems, or, as is usually the case, it may consist of the theory and principles of

processes, together with an assignment of exercises illustrative of the theory.

1. THE STUDY OF THE TEXTBOOK

Studying a mathematical textbook involves much more than the *mere reading* of the statements of principles and of the explanation of processes. The student must usually read the assigned paragraphs several times and must frequently turn back and re-read portions of the text included in previous lessons. In this manner the various points in the reasoning or explanations can be thought over, and *the habit of asking self-put questions about the work can be acquired.*

First of all, in preparing a lesson, try to find out what it is about—what its purpose is. Try to decide how you yourself would go about the accomplishment of the task and, if possible, *make an independent attempt of your own.* The more consideration you give to such an attempt, the greater scientific power you will gain.

In particular:

(A) The student should remember that the words in science have exact meanings and, of course, these meanings must be known to the student. In studying mathematics the student should acquire and use the language of mathematics. For example, he should not say "equation" when he means "expression." Indeed, he should go farther than this. He should make a conscious effort to use absolutely correct English, not only in written work but in oral work as well.

(B) While studying the text, work out theorems or illustrative examples with pen and ink. Do not rely upon a mere reading—even repeated readings—of a new piece of reasoning or of the explanation of a new process.

(C) *Read over all of the lesson assigned in the text a last time after working the assigned exercises.* The text will probably have a new meaning after working out the special cases in the exercises. This habit will give a meaning to the words, "Learn by doing."

(D) Finally, make a mental summary of each lesson.

(E) Review often.

2. THE WORKING OF EXERCISES

(F) Read each exercise or problem carefully and *plan a method of attack in advance* in order to facilitate arrangements of equations and computations and the drawing of figures.

(G) Look at your result and see if it is a reasonable one.

(H) Check result.

(I) Indicate the results by a distinguishing mark, or summarize in logical order.

(J) The figures and diagrams should have sufficient lettering, titles, etc., to make them self-explanatory. The units of measure used should, of course, be clearly indicated.

(K) Do all work neatly the first time and (except drawings) *invariably in ink*. Try to have the first draft sufficiently neat in appearance and arrangement to hand in to your instructor.

(L) After the first draft has been finished, read it over carefully to see where it may be improved in method or arrangement and *think* about the processes you have used. If small changes only are needed to effect the desired improvement, make them by drawing lines through the portions to be changed and by making neat insertions. If considerable changes are necessary, do the work over.

The study and improvement of the work will prove to be of fully as much importance to you as the doing of the work itself.

(M) See to it that each piece of work or exercise is *complete*. On any piece of written work *the nature of the problem should be clearly and briefly stated*. The student should learn to think of each piece of work as a thing that is in itself worth while. Hence each detail should be attended to before the work is submitted to the instructor. See that sufficient explanation is given and that the numbers and magnitudes are adequately named and labelled.

TO THE INSTRUCTOR

The instructor cannot insist too emphatically upon the requirement that all mathematical work done by the student—whether preliminary work, numerical scratch work, or any other kind (except drawings)—shall be carried out with pen and ink upon paper of suitable size. This should, of course, include all work done at home, irrespective of whether it is to be submitted to the instructor or not. The "psychological effect" of this requirement will be found to entrain much more than the acquirement of mere technique. If properly insisted upon, orderly and systematic habits of work will lead to orderly and systematic habits of thought. The final results will be very gratifying to those who sufficiently persist in this requirement.

At institutions whose requirements for admisstion include more than one and one-half units of preparatory algebra, nearly all of Chapters VI, VII, and VIII may be omitted from the course.

An asterisk attached to a section number indicates that the section

may be omitted. These sections will frequently be found useful in forming the basis of discussion by the instructor.

The usual one and one-half year of secondary school Algebra, including the solution of quadratic equations and a knowledge of fractional and negative exponents, is required for the work of this course. *In the appendix (Chapter XV) will be found material for a brief review of factoring, quadratics, and exponents, upon which a week or ten days should be spent before beginning the regular work in this text.*

This review chapter is placed last because the amount of material in it is greater than need be taken in all cases and also because college students do not like to be confronted on the first page of a scientific text-book with elementary work of high school grade.

GREEK ALPHABET

Capitals	Lower case	Names	Capitals	Lower case	Names
A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

MATHEMATICAL SIGNS AND SYMBOLS

...	read	*and so on.*
$\equiv$	read	*is identical with.*
$\neq$	read	*is not equal to.*
$\doteq$	read	*approaches.*
$\fallingdotseq$	read	*is approximately equal to.*
$>$	read	*is greater than.*
$<$	read	*is less than.*
$\geq$	read	*is greater than or equal to.*
(a, b)	read	*point whose coördinates are* a *and* b.
$\underline{\lfloor n}$	read	*factorial* n.
$n!$	read	*factorial* n or n *admiration.*
$\lim_{x \doteq a}\left[f(x)\right]$	read	*limit of* $f(x)$ as x *approaches* a.
$x = \infty$	read	x *becomes infinite.*
$\lvert a \rvert$	read	*absolute value of* a.
$\log_a x$	read	*logarithm of* x *to the base* a.
$\lg x$	read	*common logarithm of* x.
$\ln x$	read	*natural logarithm of* x.
$\sum_{n=1}^{n=r} u_n$	read	*summation from* $n = 1$ to $n = r$ *of* u_n.

ELEMENTARY MATHEMATICAL ANALYSIS

CHAPTER I

FUNCTIONS AND THEIR GRAPHIC REPRESENTATION

1. Quantity or Magnitude, and Number.—In algebra one is concerned with such equations as

$$y = 2x,$$
$$x + y = 3,$$
$$s = (1 + i)^n.$$

In these equations the letters i, n, s, x, y stand for **numbers,** as do the letters in any algebraic equation. The first equation, for example, states that the number y is always double the number x. Suppose, however, that one is considering a rectangle whose width is always equal to 2 in. Let the length of the rectangle be x in., and its area y sq. in. Then the relation between the area and the length of the rectangle is given by this same equation $y = 2x$. The numbers y and x now have special significance, since they are the measures of the area and the length, respectively, of the rectangle. It is important to note, however, that the equation still expresses a relationship between numbers, namely, the number of inches in the length and the number of square inches in the area. This example illustrates the distinction between **magnitude** or **quantity,** and **number.** The magnitudes or quantities in question are the length of the rectangle (x in.), and the area of the rectangle (y sq. in.); while x and y are mere numbers, or, as is sometimes said to emphasize this fact, *pure* or *abstract* numbers. Quantity or magnitude is an answer to the question: "How much?" Number is an answer to the question: "How many?" At a gas-filling station, the motorist, in answer to the question: "How much?" would say: "Ten gallons" (quantity or magnitude). In answer to

the question: "How many gallons?" he would say: "Ten" (number).

2. Constants and Variables.—In the discussion of any problem quantities may be involved whose values do not change. Such quantities are called **constants.** Quantities, on the other hand, which do not preserve the same value throughout the discussion in question are called **variables.** A quantity which is a constant has for its measure a single *fixed* or *constant* number, while a quantity which is a variable requires a varying number to measure it.

The following are variables: The distance from the center of mass of the earth to the center of mass of the sun, the steam pressure in the cylinder of a running engine, the electromotive force in an alternating current, the angles which locate the position of a planet in the heavens, the age of a man. When any of these quantities are measured, the resulting numbers are also called *variables.*

The equation $s = (1 + i)^n$ gives the number of dollars s produced by \$1 at compound interest for n years at a rate of per cent i. If, in a given problem, the interest rate be a constant, the amount s and the number of years n would be variables, and the equation gives the amount after any number of years. However if the problem be to find what times are necessary for money to double at various rates, then i and n are the variables, and s is a constant whose value is 2.

The equation $PV = RT$, where P is the pressure, V the volume, and T the absolute temperature of a given mass of gas, and R is a constant whose value depends upon the amount and the kind of the gas, is called the gas law. In a given experiment, the pressure may be held constant, whereupon V and T become the variables. If the temperature be held constant, P and V become the variables.

It is clear from the above examples that a quantity may be a constant in one problem and a variable in another. There are, however, quantities (such as the ratio of the diameter of a circle to its circumference, the charge of an electron, the universal constant of gravitation, etc.) whose values are the same in all conceivable problems. Such quantities are called **absolute constants.**

The following are absolute constants: The electrical resistance of pure copper at 60°F., the combining weight of oxygen, the density of pure iron, the velocity of light in empty space.

In the equation $s = (\frac{1}{2})gt^2$ (the law of falling bodies) is the constant g an absolute constant?

3. Functions.—Goods sent by freight are divided into first, second, third, fourth, and fifth classes. If a package of given weight is to be shipped from, say, Chicago to New York, the charge for freight depends upon the class of the shipment. On account of this dependence, the freight charge is said to be a **function** of the class of the shipment. In the same way, the freight charge is a function of the weight of the package. It is also a function of the distance carried.

When the radius of a circle is known, the area is determined, and, on account of this relationship between the area and the radius, the area is said to be a function of the radius. In this case the relationship may be expressed by means of an equation, namely the equation $A = \pi r^2$, where A is the number of square inches in the area, and r the number of inches in the radius.

In general, whenever a dependence exists between two quantities, and thus also between the numbers which measure these quantities, one quantity or one number is said to be a function of the other. This is made more precise in the following **definition:**

Any number u is said to be a **function** of another number t if, when the value of t is given, the value of u is determined.

This definition covers the case of a functional relationship between two variables. Often the value of a number u is determined not by the value of a single other number t, but rather by the value of several other numbers r, s, t, etc. In this more general case the number u is said to be a function of the numbers r, s, t, etc. In the first example given above, for instance, the freight charge is determined by the class and the weight of the package, and the distance carried, and is said to be a function of these last three quantities.

The length of a rod is a function of its temperature. The area of a square is a function of the length of a side. The square root of a number is a function of the number. The strength of an iron rod is a function of its diameter. The pressure in the ocean is a function of the depth below the surface. The price of a railroad ticket is a function of the distance to be traveled. The temperature Fahrenheit is a function of the temperature Centigrade.

Expressions such as $x + 2$, or $3x^2 - 6x + 2$ are functions of x, since, when x is given, the values of these expressions are deter-

mined. Indeed, any mathematical expression is, by the above definition, a function of the letter or letters that occur in it. Each of the expressions,

$$t^2 + 4t + 1,$$
$$\frac{t-1}{2t+2},$$
$$\sqrt{t+4} + t^2 - \frac{3}{t},$$

is a function of t.

When one number is a function of another, it is not always possible or convenient to express their dependence by means of an equation. It has been seen that the freight charge is a function of the class and the weight of the package, and of the distance it is to be carried. Only the second of the functional relations just named can readily be expressed by means of an algebraic equation. *The definition of the function is given (for any particular railroad) by the complete tariff book of the railroad.*

The fee charged for a money order is a function of the amount of the order, but here again it is not convenient to represent this dependence by means of an equation. The following table defines this functional relationship for money orders of less that $100:

TABLE I.—FEES CHARGED FOR MONEY ORDERS

Amount of money order in dollars	Fee in cents
0–2.5	3
2.5–5.0	5
5–10	8
10–20	10
20–30	12
30–40	15
40–50	18
50–60	20
60–75	25
75–100	30

This table does not, however, completely define the function, for it does not tell what fee is to be charged for such amounts as \$2.5, \$5, etc., which are division points in the table. One can define the function for these values by the supplementary statement (for example): "For the critical amounts always charge the higher fee." As a matter of fact, however, the lower fee is charged.

Functional relationships may be expressed by verbal statements, by equations, by graphs drawn by a recording instrument, by graphs drawn from equations or data, and by numerical tables. The last-mentioned method is of special importance in applied mathematics. Any set of data obtained experimentally, for example, defines a functional relationship in this way.

Illustration 1.—One side of a rectangle is 2 cm. The other side is $(x + 2)$ cm. Express the area A of the rectangle as a function of x.

The area is the product of the breadth by the length of the rectangle. Hence

$$A = 2(x + 2) = 2x + 4, \tag{1}$$

which is the function of x sought.

Illustration 2.—The hypotenuse of a right triangle is 10 in. One side is x in. Express A, the area of the triangle, as a function of x.

Since the hypotenuse squared equals the sum of the squares of the two legs, we may write

$$10^2 = x^2 + y^2, \tag{1}$$

where y stands for the length in inches of the second leg of the triangle. But we know that

$$A = \tfrac{1}{2}xy. \tag{2}$$

From Eq. (1)

$$y = \sqrt{10^2 - x^2}. \tag{3}$$

Substituting in Eq. (2), we have

$$A = \tfrac{1}{2}x\sqrt{100 - x^2}, \tag{4}$$

which is the function of x desired.

Illustration 3.—Express the amount A of \$1 at simple interest at 6 per cent for n years as a function of n.

The interest on \$1 for n years equals $\$\frac{6}{100}n$. Hence the amount (which is the principal plus the interest) is expressed by

$$A = 1 + \tfrac{6}{100}n.$$

4. Explicit and Implicit Functions.—In the above illustrations, when the problem has been to express some number A as a function of another number x, the result has been an equation whose

left member is A, and on the right side of which no variable other than x appears. When the relationship between A and x is so expressed, A is said to be an **explicit** function of x, and x is said to be the **argument** of the function. Thus in the equation

$$A = 2x^2 + 5$$

A is an explicit function of the argument x. This equation can, however, be solved for x, so as to read

$$x = \tfrac{1}{2}\sqrt{2A - 10},$$

and x is now an explicit function of the argument A. The equation may also be written

$$A - 2x^2 - 5 = 0.$$

Here neither variable is expressed explicitly as a function of the other; but, since the relation implies that A is a function of x (or x a function of A), it is said that A is here an **implicit** function of x (or x an implicit function of A).

It should be understood that, although in the first of the three above equations one calls A an explicit function of x, and, in the third equation, A an implicit function of x, the functional relationship between A and x is the same in the two cases. In the first case it is expressed explicitly; in the other, implicitly.

Exercises

In the following exercises the function described can be represented by a mathematical expression. The problem is to set up the expression in each case, writing one variable as an explicit function of the other unless specifically directed otherwise.

1. One side of a rectangle is 10 ft. Express the area A as a function of the other side x.

2. One leg of a right triangle is 15 ft. Express the area A as a function of the other leg x.

3. The base of a triangle is 12 ft. Express the area as a function of the altitude l.

4. Express the circumference of a circle as a function (a) of its radius r; (b) of its diameter d.

5. Express the diagonal d of a square as a function of one side x.

6. One leg of a right triangle is 10. Express the hypotenuse h as a function of the other leg x.

7. A ship B sails on a course AB perpendicular to OA. If $OA = 30$ miles, express the distance of the ship from O as a function of AB.

8. A circle has a radius 10 units. Express the length of a chord as a function of its distance from the center.

9. An isosceles triangle has two sides each equal to 15 cm., and the third side equal to x cm. Express the area of the triangle as a function of x.

10. A right cone is inscribed in a sphere of radius 12 in. Express the volume of the cone as a function of its altitude l.

Hint: The distance from the center of the sphere to the base of the cone is $(l - 12)$, if $l > 12$. The radius of the base of the cone is $\sqrt{12^2 - (l - 12)^2}$ or $\sqrt{24l - l^2}$. What if $l < 12$?

11. A right cone is inscribed in a sphere of radius a. Express the volume of the cone as a function of its altitude l.

12. In Exercises 1 and 2, express x as a function of A.

13. In Exercise 6, express x as a function of h.

14. The volume V of a cube is a function of the length x of a side. Write the equation which expresses one of these variables as an implicit function of the other.

5. Functional Notation.—When it is desired to express the fact that y is a function of x, this can be done, of course, by the statement

$$y \text{ is a function of } x,$$

but it is usually convenient to abbreviate this expression and use the notation

$$y = f(x),$$

this equation being defined to mean the same thing as the statement just above; in fact, the equation should be read "y is a function of x." If in a certain problem one has a number u which is a function of another number t, and a third number v which is also a function of t, it would be misleading to indicate these facts by writing

$$u = f(t); v = f(t),$$

since the notation might be misinterpreted as meaning that u and v were the same function of t, the same notation being used in the two cases. Therefore, the functional notation is extended, and one writes, for example,

$$u = f_1(t); v = f_2(t),$$

which can be read "u is the f_1-function of t; v is the f_2-func-

tion of t" and the confusion suggested above is avoided. The functional notation is thus a combination of the name of the function, "the f-function," and the name of the argument, x. Other symbols commonly used to express functions of x are

$$\phi(x),\ X(x),\ f'(x),\ F(x),\ \text{etc.},$$

which may be read the "ϕ-function of x," the "X-function of x," etc., or, more briefly, the "ϕ of x," the "X of x," etc.

Expressing the fact that temperature reading Fahrenheit (F) is a function of temperature reading Centigrade (C), we may write:

$$F = f(C).$$

This equation states that there is a functional relationship between F and C. The relationship itself is expressed by means of the equation

$$F = (\tfrac{9}{5})C + 32.$$

The fact that the charge for freight is a function of class, weight, and distance may be indicated by writing

$$r = f(c, w, d).$$

To make this functional symbol explicit, that is, to tell just what function r is of c, w, and d, might require that we be furnished with the complete schedule as printed in the freight tariff book of the railroad. The dependence of the charge upon class and weight can usually be readily expressed, but the dependence upon distance often contains arbitrary elements that cause it to vary irregularly, even on different branches of the same railroad. A complete specification of the functional symbol f would be considered given in this case when the tariff book of the railroad was in hand.

If $f(x) = x^2 - 2x + 5$, then the notation $f(2)$ indicates that the special value 2 is to be substituted for the general argument x. That is, $f(2) = 2^2 - 2 \cdot 2 + 5 = 5$.

Exercise

If $f(x) = x^3 + 2x^2 - x + 3$, what are $f(1)$, $f(-2)$, $f(0)$, $f(5)$, $f(a)$? Note especially that $f(0)$ is not zero.

6. The Graphic Representation of Functions. Scales.—Great simplicity and clearness are often gained, in the study of functional

relationships, by making use of some sort of a pictorial or graphic representation of the relationship. Practically all of the many schemes of representation make use of some sort of a scale.

Select a series of points along any curve and mark the points of division with the numbers of any sequence.[1] The result of such a construction is called a **scale.** Thus, in Fig. 1 the points along the curve OA have been selected and marked in order with the numbers of the sequence:

$$0, \tfrac{1}{4}, \tfrac{1}{2}, 1, 2, 2\tfrac{1}{2}, 3, 5, 7, 8.$$

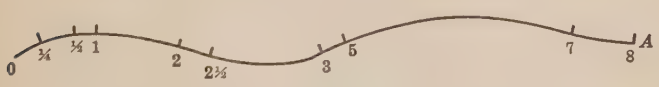


FIG. 1.—A non-uniform scale.

Thus, primitive man might have made notches along a twig and then made use of it in making certain measurements of interest to him. If such a scale were to become generally used by others, it would be desirable to make many copies of the original scale. It would, therefore, be necessary to use a twig whose shape could be readily duplicated, such, for example, as a straight stick; and it would also be necessary to attach the same symbols invariably to the same divisions.

Certain advantages are gained (often at the expense of others, however) if the distances between consecutive points of division are kept the same, that is, when the intervals are laid off by repetition of the same selected distance. When this is done, the scale is called a **uniform scale.** Primitive man might have selected for such uniform distance the length of his foot or sandal, the breadth of his hand, the distance from the elbow to the end of the middle finger (the cubit), the length of a step in pacing (the yard), the amount he could stretch with both arms extended (the fathom), etc.

We are familiar with many scales, such as those seen on a yardstick, the dial of a clock, a thermometer, a sundial, a steam gage, an ammeter or voltmeter, the arm of a storekeeper's scales, etc. The scales on a clock, a yardstick, or a steel tape are uni-

[1] A sequence of numbers here means a set of numbers arranged in order of magnitude.

form. Those on a sundial, on some ammeters, or on a good thermometer are not uniform.

One of the most important advantages of a uniform scale is the fact that the place of beginning, or zero, may be taken at any one

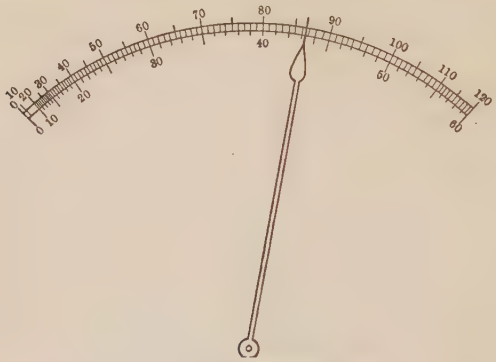


Fig. 2.—An ammeter scale.

of the points of division. This is not true of a non-uniform scale. If a sundial be not properly oriented, it is useless. If the needle of an ammeter be bent, the instrument cannot be used. It is always necessary in using such an instrument to know that the zero is

Fig. 3.—Sun-dial scale.

correct. If, however, a yardstick or a steel tape be broken, it may still be used for measuring lengths. The student may think of many other advantages gained in using a uniform scale.

7. Formal Definition of a Scale.—If points be selected in order along any curve corresponding, one to one, to the numbers of any sequence, the curve, with its divisions, is called a **scale.**

The notion of **one-to-one correspondence,** included in this definition, is frequently used in mathematics.

In mathematics we frequently speak of the **arithmetical scale** and of the **algebraic scale.** The arithmetical scale corresponds to the numbers of the sequence

$$0, 1, 2, 3, 4, 5, \ldots$$

and such intermediate numbers as may be desired. It is usually

Fig. 4.—A uniform arithmetical scale.

represented by a uniform scale as in Fig. 4. The algebraic scale corresponds to the numbers of the sequence

$$\ldots -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \ldots$$

and such intermediate numbers as may be desired. It is usually

Fig. 5.—A uniform algebraic scale.

represented by a uniform scale as in Fig. 5. The arithmetical scale begins at 0 and extends indefinitely in one direction. The algebraic scale has no point of beginning; the zero is placed at any desired point and the positive and the negative numbers are then attached to the divisions to the right and the left, respectively, of the zero so selected. The algebraic scale extends indefinitely in both directions.

Exercises

1. On a uniform algebraic scale, how far is the point marked 5 from the point marked 7? How far is the point marked 6 from the point marked 10.5? How far is the point marked -10.3 from the point marked 13.6?

2. Show that the distance between two points selected anywhere on the uniform algebraic scale is always found by subtraction.

3. What points of the uniform algebraic scale are distant 5 from the point 3 of that scale? What point of the uniform arithmetical scale is distant 5 from the point 3 of that scale?

4. If two algebraic scales intersect at right angles, the common point being the zero of both scales, explain how to find the distance from any point of one scale to any point of the other scale.

FIG. 6.—Centimeter-inch double scale.

FIG. 7.—Fahrenheit-centigrade scale.

FIG. 8.

8. Two Uniform Scales in Juxtaposition, or Double Scales.— The relation between two magnitudes or quantities, or between two numbers, may be shown conveniently by placing two scales side by side. Thus, the relation between the number of centimeters and the number of inches in any length may be shown by placing a centimeter scale and a foot rule side by side with their zeros coinciding as in Fig. 6. From this figure it is seen that 1 in. corresponds to 2.54 cm.; 3.3 in. to 8.38 cm.; 4.6 in. to 11.68 cm.; that 5 cm. correspond to 1.97 in.; 8.5 cm. to 3.34 in., etc.

A thermometer is frequently seen bearing both Fahrenheit and Centigrade scales (see Fig. 7). It is obvious that the double scale of such a thermometer may be used (within the limits of its range) for converting any temperature reading Fahrenheit into the corresponding Centigrade equivalent or *vice versa*. From Fig. 7 it is seen that 72°F. corresponds to 22.2°C., 212°F. to 100°C., 32°F. to 0°C.; that 21°C. corresponds to 69.8°F., 72°C. to 161.6°F.

The construction of scales of the kind considered above may be made to depend upon the following problem in elementary geometry: *To divide a given line into a given number of equal parts.*

Illustration.—In Fig. 9 is given a double scale *OA-OB* showing the correspondence between speed expressed in miles per hour and speed expressed in feet per second. The student will reproduce neatly and accurately the drawing, on a larger scale, in accordance with the directions given below.

A mile contains 5280 ft., an hour contains 3600 sec. Hence, 1 mile per hour equals 5280/3600, or 22/15 ft. per second. Therefore if two uniform scales (Fig. 9), one representing speed expressed in feet per second and the other representing speed expressed in miles per hour, be constructed with their zeros coinciding and with the point marked 22 of the first coinciding with the point marked 15 of the second, the double scale may be used for converting speed expressed as miles per hour into speed expressed as feet per second, or *vice versa*.

Lay off with a scale a line *OA* 11 in. long. Divide this line into 22 equal parts, and subdivide each division into 2 equal parts. Mark these divisions and subdivisions as indicated in Fig. 9. Draw the line *OK*, making the angle *KOB* about 30°. With a pair of bow dividers or with a scale lay off on *OK* 15 equal divisions, about $\frac{3}{4}$ in. each. Let the last point of division be marked *C*. *OC* is then divided into 15 equal parts. Draw *CA*. With a pair of

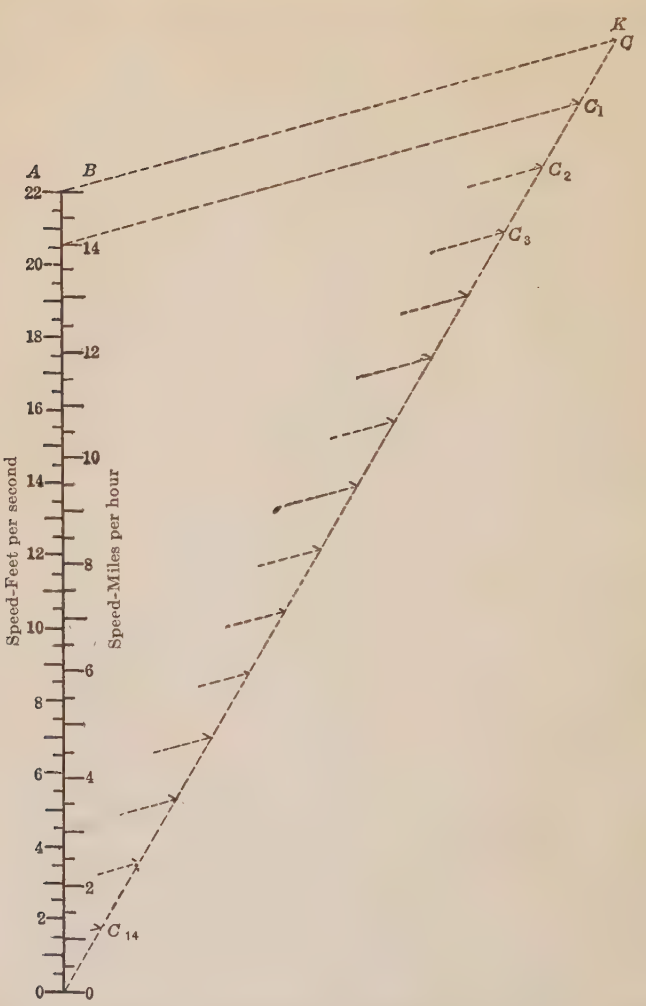


FIG. 9.—Method of construction of double scale showing relation between speeds as measured in miles per hour and in feet per second.

triangles draw lines through the points C_1, C_2, C_3, . . . C_{14} parallel to CA, intersecting the line OA in the points marked 15, 14, 13, . . . 1, respectively. Why is OB a uniform scale divided into 15 equal parts?

Mark the scales OA and OB in red ink with a new set of numbers so that the double scale may also be used for converting speeds if the readings fall between 15 and 30 ft. per second instead of between 0 and 15.

From the double scale just constructed, find the speeds expressed as miles per hour corresponding to speeds of 2, 4, 5, 11, 14, 20, and 25 ft. per second.

The lengths selected to represent the various units in any diagram are, of course, arbitrary. As, however, the student is expected to prepare the various constructions and diagrams required for the exercises in this book on paper of standard letter size (that is, 8½ by 11 in.), the units selected should be such as to permit a convenient and practical construction upon sheets of that size.

Exercises

The student is expected to carry out the actual construction of only one of the double scales described in the following exercises.

Before making one of these scales, the student should read the remarks numbered 4, 5, and 6, Sec. **14.**

1. Draw a double scale showing the relation between pressure expressed as inches of mercury and as feet of water, knowing that the density of mercury is 13.6 times that of water.

These are two of the common ways of expressing pressure. Water pressure at water-power plants, and often for city water service, is expressed in terms of head in feet. Barometric pressure, and the vacuum in the suction pipe of a pump and in the exhaust of a condensing steam engine, are expressed in inches of mercury. The *approximate relations* between these units, *i.e.*, 1 atm. = 30 in. of mercury = 32 ft. of water = 15 lb. per square inch, are known to every student of elementary physics. To obtain, in terms of feet of water, the pressure equivalent of x ft. of mercury, multiply x by 13.6, the specific gravity of mercury. This product divided by 12, or $1.13x$, gives the number of feet of water corresponding to x in. of mercury.

If we let the scale of inches of mercury range from 0 to 10, then the scale of feet of water must range from 0 to 11.3. Hence draw a line OA 10 in. long divided into inches and tenths to represent inches

of mercury. Draw any line OC through O and lay off on it 11.3 uniform intervals (inch intervals will be satisfactory). Connect the end division on OA with the end division on OC by a line AC. Then from 1, 2, 3, . . . in. on OC draw parallels to AC, thus forming adjacent to OA the scale of equivalent feet of water. Each of these intervals can then be subdivided into 10 equal parts corresponding to tenths of feet of water.

2. Draw a double scale showing pressure expressed as feet of water, and as pounds per square inch, knowing that 1 cu. ft. of water weighs 62.5 lb.

The weight of 1 cu. ft. of water, 62.5 lb., divided by 144, the number of square inches on one face of a cubic foot, gives 0.434 lb. per square inch as the equivalent of 1 ft. of water pressure.

One pound per square inch is equivalent, therefore, to 1/0.434 or 2.30 ft. of water pressure. If we let the scale of pounds range from 0 to 10, we may select 1 in. as the equivalent of 1 lb. per square inch, and divide the scale OA into inches and tenths to represent this magnitude. Draw OC through O, and lay off 23 uniform intervals on OC, ½ in. being a convenient length for each of these parts. Connect the end division of OC with A and through all points of division of OC draw lines parallel to CA. The range may be extended to any amount desired by annexing ciphers to the numbers attached to the two scales.

Extending the range by annexing ciphers to the attached numbers is obviously practicable so long as the various intervals or units are decimally subdivided. The method is impracticable for scales that are not decimally subdivided, such as shillings and pence, degrees and minutes, feet and inches, etc.

3. Draw a double scale showing the relation between cubic feet, and gallons. One gallon equals 231 cu. in., but use the approximate relation, 1 cu. ft. equals 7½ gal. Divide the scale of cubic feet into tenths, the scale of gallons into fourths to correspond to quarts.

It is obvious that it is always necessary first to select the range of the various scales, but it is quite as well in this case to show the equivalents for 1 cu. ft. only, as numbers on the various scales can be multiplied by 10, 100, or 1000, etc., to show the equivalents for larger amounts.

Select 10 in. = 1 cu. ft. for the scale (OA) of cubic feet. Draw the line OC. On OC lay off 7½ equal parts (say, 7½ in.). Connect the end division with A and draw the parallel lines exactly as with previous examples. The intervals of the scale of gallons can then be subdivided into the 4 equal parts to show quarts.

4. Draw a double scale showing the relation between cubic feet and liters. One cubic foot equals 28⅓ liters.

5. If a double scale be drawn on a deformable body, as, for example, on a rubber band, would the double scale still represent true relations when the rubber band is stretched? What if the stretching were not uniform?

6. From Fig. 9, find the number of miles per hour corresponding to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, and 15 ft. per second. Place the results in tabular form, *i.e.*, in the first of two adjacent vertical columns place the numbers 0, 1, 2, . . . 15; opposite these numbers place in the second vertical column the corresponding numbers representing speed as miles per hour. Give the first vertical column the heading "Speed-ft./sec.," and the second column the heading "Speed-mi./hr." As the speed changes from 1 ft. per second to 2 ft. per second, the speed changes by what amount in miles per hour? As the speed changes from 3 ft. per second to 4 ft. per second, the speed in miles per hour changes by what amount? The change in speed as miles per hour is how many times the change in the speed as feet per second?

9. Non-uniform Scales. Function Scales.—Each scale of the double scales constructed in the preceding section was a uniform scale. The construction of a double scale of this kind was possible because the change in the number of units of one magnitude represented was directly proportional to the corresponding change in the number of units of the other magnitude. In a uniform scale, the distance from the division point marked 0 to, say, the division point marked 3 is 3 times the distance from the point marked 0 to the point marked 1. That is, the numbers attached to the various division points represent the actual distances to these points from the point marked 0, using the distance from 0 to 1 as a unit of distance. Consider, however, the double scale shown in

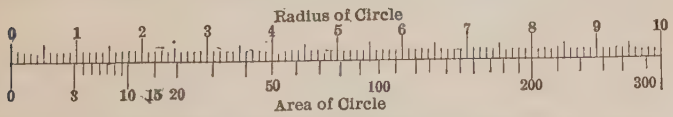


FIG. 10.—Double scale showing the relation between the area of a circle and its radius.

Fig. 10. This double scale represents the relation between the radius and the area of a circle, the radius being represented by

points on the upper uniform scale, and the area by points on the lower non-uniform scale. This relationship cannot be represented by two uniform scales, because a change in area is proportional to the change in the square of the radius, and not to the change in the radius itself. If the distance from 0 to 1 on the uniform scale be adopted as a unit of length, then we may say that the number attached to any division point on the non-uniform scale is π times the square of the distance of that division point from 0, the relation between the radius and the area of a circle being given by the equation $A = \pi r^2$. The non-uniform scale furnishes a graphic representation of the functional relationship,

$$A = f(r) = \pi r^2,$$

and, by interpolation on both the uniform and the non-uniform scales, the value of A which corresponds to any particular value of r between 0 an 10 may be read. Such a scale as the non-uniform scale here shown is called a **function scale,** it being, in this particular case, a function scale for the function $f(r) = \pi r^2$.

A function scale for the function $y = 1/x$ would be constructed by choosing a unit of length and an initial point marked 0. Then the number attached to any other division point is the reciprocal of the number which measures the distance of that point from 0. Figure 11 shows the result of this construction. This scale gives

FIG. 11.—Function scale for the function $1/x$.

a picture of the functional relationship $y = 1/x$. If this scale is to be used in reading off values of the function $1/x$, it is necessary, as in Fig. 10, to place in contact with it, and with the zero points coinciding, a uniform scale employing the same unit of distance.

When two variables are proportional one to the other, this functional relationship is represented by a particularly simple sort of function scale, namely (as we have seen), by uniform scales. Thus, the double scales of Figs. 6, 7, and 8, as well as the non-uniform scales of Figs. 10 and 11, are function scales.

The representation of functional relationships by means of function scales is not a particularly useful one to the pure mathe-

matician, but is of great interest and usefulness in applied mathematics. The engineer's slide rule, as will be seen later, is a combination of such function scales. Engineers, physicists, chem-

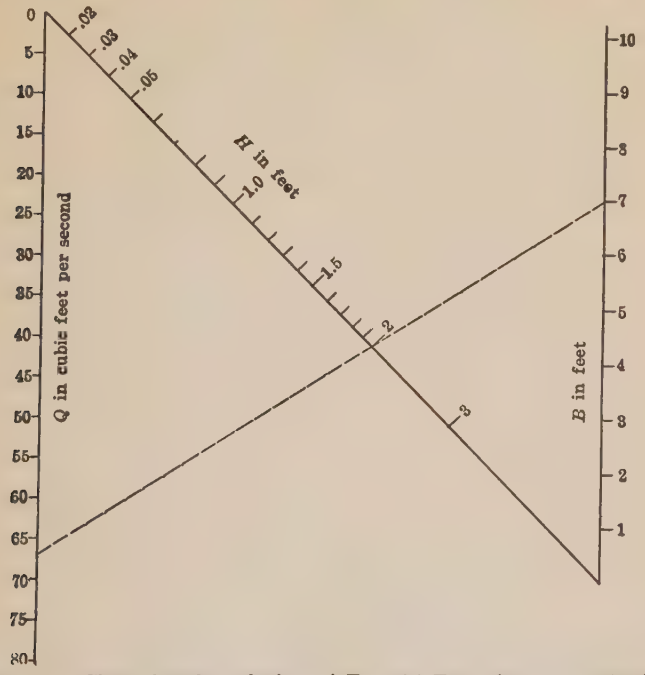


FIG. 12.—Chart for the solution of Francis' Formula $Q = 3.33BH^{3/2}$. This formula is often used to calculate the flow Q, in cubic feet per second, over a weir of breadth B feet, H being the head, in feet, of water on the weir. If a straight line be drawn joining any two of these three quantities, the third quantity is given by the intersection of this straight line with the third scale. For example, the dotted line drawn in the figure shows that if there be a head of 2 feet on a weir of breadth 7 feet, the flow is approximately 67 cubic feet per minute.

ists, etc. make extensive use of charts and diagrams, called alignment charts and nomographs, in the construction of which function scales play a fundamental rôle. Figure 12 shows an example of such an alignment chart.[1]

[1] Figure 12 is reproduced with permission from "Design of Diagrams for Engineering Formulas," by Hewes & Seward, McGraw-Hill Book Co., 1924.

10. Coordinate Systems.—It has been seen above that a functional relationship may be exhibited by means of a function scale. It is often convenient to use a somewhat different mode of representation, which makes use of what is called a **system of coordinates.** By definition it follows that when $y = f(x)$, to a definite value of x there corresponds a definite value of y. This pair of corresponding numbers may be used in a variety of ways to locate a point in a plane. If another value of x is now chosen, and the corresponding value of y obtained, this second pair of numbers locates a second point in the plane. If a large number of such pairs of values are computed, and the points which they determine are located, a curve can be drawn through these points and a picture of the function thus obtained. Thus, any scheme whereby

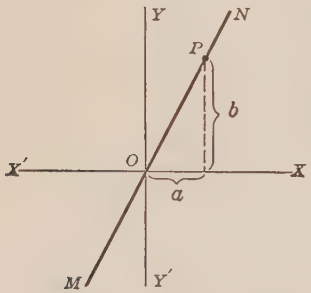


Fig. 13.

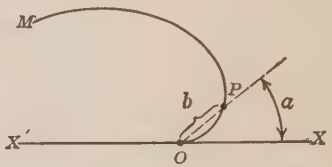


Fig. 14.

Figs. 13 and 14.—Two representations of the relationship "the number b is always twice the number a."

a point in a plane is located by a pair of numbers furnishes the basis for a graphic representation of the function. One such scheme, for example, would consist of choosing a fixed reference point O in the plane, and drawing a vertical and horizontal line through O. Then one number, say x, would be taken as the distance of the located point to the right of the vertical line (to the left if x be negative), and the other number, y, the distance up from the horizontal line (down if y be negative). Another scheme would consist of using one number as the distance from a fixed initial point O to the located point P, and the other as the number of degrees between the line OP and some fixed line through O—say a line bearing due east. Figures 13 and 14 furnish representations, using the two schemes just suggested, of the functional relationship

contained in the statement "the number b is always double the number a." In Fig. 13 each pair of corresponding values of a and b locates a point which is a units (say, a in.) to the right and b units (say, b in.) above the reference point O. In Fig. 14 each pair of corresponding values of a and b locates a point which is b units of distance from O, the angle between OP and the fixed reference line OO' being a degrees. In order that the figure include enough of the curve to illustrate the idea, in measuring the distance from O to a point on the curve, an inch is taken equal to 200 units of distance. The student should study these two figures until he sees clearly that, although their appearance is quite different, they are representations of the same functional relationship, one portraying the same facts as the other. If, in Fig. 13 a centimeter instead of an inch were used as a unit of distance, would the figure need to be altered in any way? If, in Fig. 14, an inch equal 100 units of distance, rather than 200, how would the figure be altered?

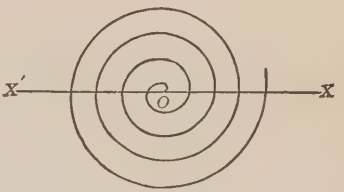


FIG. 15.—Representation of the relationship "the number b is always twice the number a."

The first of these two schemes, with which the student is already somewhat familiar from his graphic work in elementary algebra, makes use of **rectangular** or **Cartesian** coordinates, and will be explained in greater detail in the next section. The second scheme makes use of **polar** coordinates, and will be studied and used later in the book.

Figure 15 is a redrawing of Fig. 14, using a new scale (1 inch = 1000 units) for the measurement of the distance from the origin O to a point on the curve. A comparison of Figs. 13 and 15 shows, even more clearly than does a comparison of Figs. 13 and 14, that the two types of representation state the same thing; namely, that, when a increases by a fixed amount, b increases by a fixed amount. How is this shown on Fig. 15?

11. Rectangular Coordinates.—The basis of a system of rectangular coordinates is a pair of algebraic scales, intersecting at right angles at their common zero point. To locate a given

point in the plane it is merely necessary to give, in a suitable unit of measure (as centimeter, inch, etc.), the distance of the point to the right or the left of the vertical scale and its distance above or below the horizontal scale, using one scale to measure distance perpendicular to the other. Thus the point P_1 in Fig. 16 is $2\frac{1}{2}$ units to the right and $3\frac{1}{2}$ units above the standard scales. P_2 is 3 units to the left and 2 units above the standard scales, etc. Of course, these directions are to be given in mathematics by the

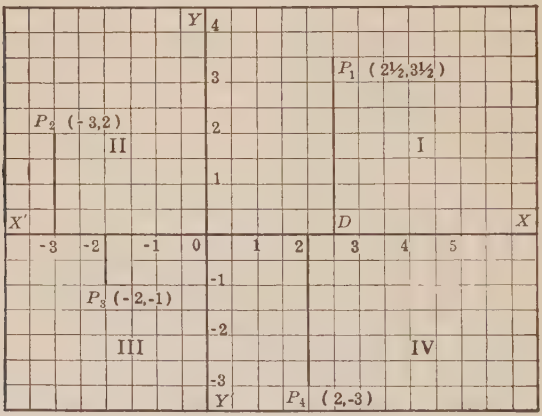


Fig. 16.—Rectangular coördinates.

use of the signs "+" and "−" of the algebraic scales, and not by the use of the words "right" or "left," "up" or "down."

It has become customary to letter one of the scales XX', called the X-**axis,** and to letter the other YY', called the Y-**axis.** In the standard case these are drawn to the right and left, and up and down, respectively, as shown in Fig. 16. The distance of any point from the Y-axis, measured parallel to the X-axis, is called the **abscissa** of the point. The distance of any point from the X-axis, measured parallel to the Y-axis, is called the **ordinate** of the point. Collectively, the abscissa and the ordinate are spoken of as the **coordinates** of the point. **Abscissa** corresponds to the **longitude**

and **ordinate** corresponds to the **latitude** of the point, referred to the X-axis as equator, and to the Y-axis as standard meridian. In the standard case, abscissas measured to the right of YY' are reckoned positive, those to the left, negative. Ordinates measured up are reckoned positive, those measured down, negative.

Rectangular coordinates are frequently called **Cartesian coordinates,** because they were first introduced into mathematics by René Descartes (1596–1650).

The point of intersection of the axes is lettered O and is called the **origin.** The four quadrants, XOY, YOX', $X'OY'$, $Y'OX$, are called the *first, second, third,* and *fourth* **quadrants,** respectively.

A point is designated by writing its abscissa and its ordinate in parentheses and in this order: Thus, (3, 4) means the point whose abscissa is 3 and whose ordinate is 4. Likewise $(-3, 4)$ means the point whose abscissa is -3 and whose ordinate is $+4$.

Abscissas are usually represented by the letter x and ordinates are usually represented by the letter y. Thus, the point whose abscissa is 3 and whose ordinate is 4, may be described as the point (3, 4), or equally well as the point $x = 3$, $y = 4$.

Unless the contrary is explicitly stated, the scales of the coordinate axes are assumed to be *straight* and *uniform* and to intersect at right angles. Exceptions to this are not uncommon.

Exercises

On suitable squared paper, select and mark a horizontal line as the X-axis (or axis of abscissas) and select and mark a vertical line as the Y-axis (or axis of ordinates). Select and mark a suitable unit of measure on each axis, for example, 1 cm. or ½ in. Then locate the points whose coordinates are given in the following exercises.

1. Draw the coordinate axes on squared paper and locate the points (3, 3), (2, 2), (1, 1), (0, 0), $(-1, -1)$, $(-2, -2)$, $(-3, -3)$.

2. Draw the axes and locate the points (2, 3), $(-2, 3)$, $(-2, -3)$, $(2, -3)$.

3. Draw the coordinate axes and locate the points (5, 0), (4, 3), (3, 4), (0, 5), $(-3, 4)$, $(-4, 3)$, $(-5, 0)$, $(-4, -3)$, $(-3, -4)$, $(0, -5)$, $(3, -4)$, $(4, -3)$.

4. Draw suitable axes and locate the points $(-3, -5)$, $(-2, -3)$, $(-1, -1)$, (0, 1) (1, 3), (2, 5), (3, 7), (4, 9).

A brief way of describing a set of points is to place the abscissas and the ordinates in tabular form, indicating abscissas by the letter x and indicating ordinates by the letter y, as follows:

x	-3	-2	-1	0	1	2	3	4
y	-5	-3	-1	1	3	5	7	9

12. Mathematical Graphs. The Locus of an Equation.— Instead of the expressions "*abscissa of a point*," or "*ordinate of a point*," it has become usual to speak merely of the **"x of a point,"** or of the **"y of a point,"** since these distances are conventionally represented by the letters x and y, respectively. If we impose certain conditions upon x and y, then it will be found that we have, by that very fact, restricted the possible points of the plane located by them to a certain array, or set of points, and that all other points of the plane fail to satisfy the conditions or restrictions imposed.

It is obvious that the command, "Find the place whose latitude equals its longitude," does not restrict or confine a person to a particular place or point. The places satisfying this condition are unlimited in number. We indicate all such points by drawing a line bisecting the angles of the first and third quadrants; at all points on this line latitude equals longitude. We speak of this line as the **locus** of the point satisfying the conditions. We might describe the same locus by saying "the y of each point of the locus equals the x," or, with the maximum brevity, simply write the equation "$y = x$." The equation "$y = x$" is called the **equation of the locus,** and the line is called the **locus of the equation.**

It is of the utmost importance to be able readily to interpret any condition imposed upon, or, what is the same thing, any relation between, variables, when these are given in words. It will greatly aid the beginner in mastering the concept of what is meant by the term *function* if he will try to think of the *meaning in words* of the relations commonly given by equations, and *vice versa*. The very elegance and brevity of the mathematical expression of relations by means of equations tends to make work with them formal and mechanical unless care is taken by the beginner to express in words the ideas and the relations so briefly expressed by the equa-

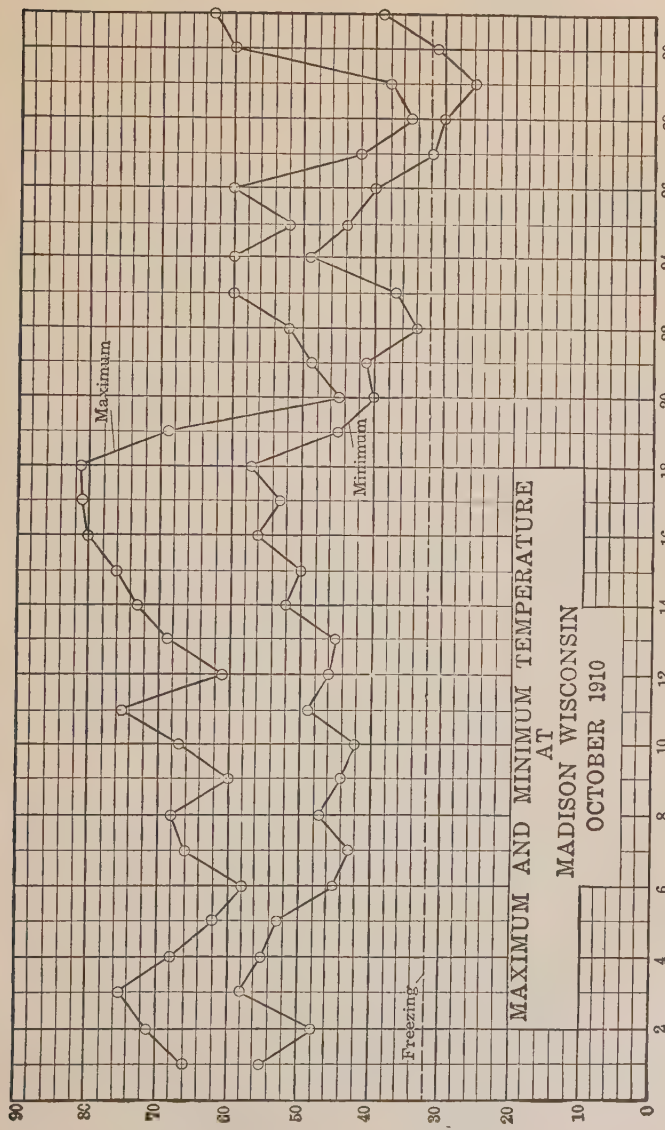


FIG. 17.—Maximum and minimum temperature at Madison, October, 1910.

tions. Unless expressed in words, the ideas are liable not to be expressed at all.

The **equation of a curve** is an equation satisfied by the coordinates of every point of the curve and by the coordinates of no other point.

The **graph of an equation** is the locus of a point (that is to say, the totality of all points) whose coordinates satisfy the equation.

13. Non-mathematical or Statistical Graphs.—From work in elementary algebra the student is familiar with the construction of statistical graphs similar to Figs. 17 and 18. The student should

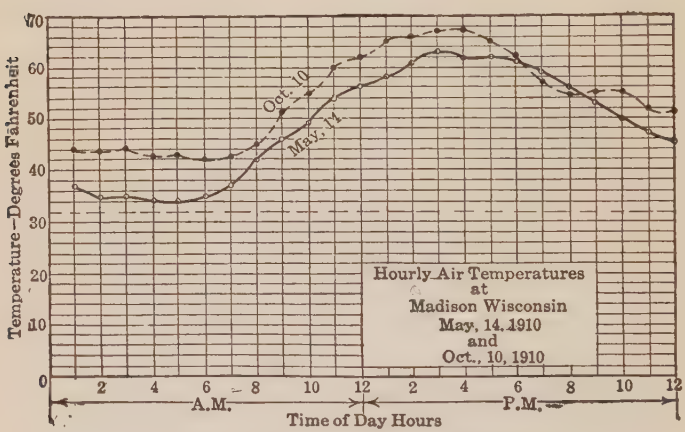


FIG. 18.—Hourly air temperatures.

carefully study the construction of these two graphs. In Fig. 17, the point at the center of any small circle represents the maximum temperature (or the minimum temperature) on a particular day. This circle is joined by a straight line to the circle representing the maximum (or minimum) temperature on the next day, and so on. The lines joining the circles enable the eye to follow at a glance the changes of temperature for the entire month. However, a point on a line between two circles has no meaning, because a point on the horizontal scale between two consecutive points has no meaning, for, of course, there is but *one* daily maximum for each day. The student should especially note that the ratio of the

distance on the horizontal scale representing days to the distance on the vertical scale representing a change of 1° in temperature

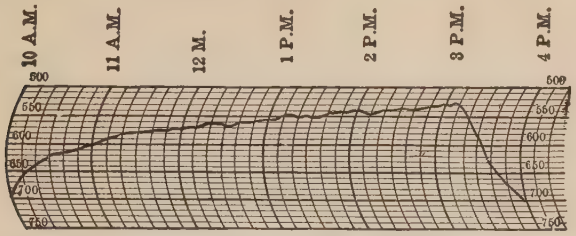


FIG. 19.—Barograph taken during a balloon journey. The vertical scale is atmospheric pressure in millimeters of mercury.

is so chosen as to make the fluctuations in the temperatures stand out prominently. In constructing statistical graphs, the student

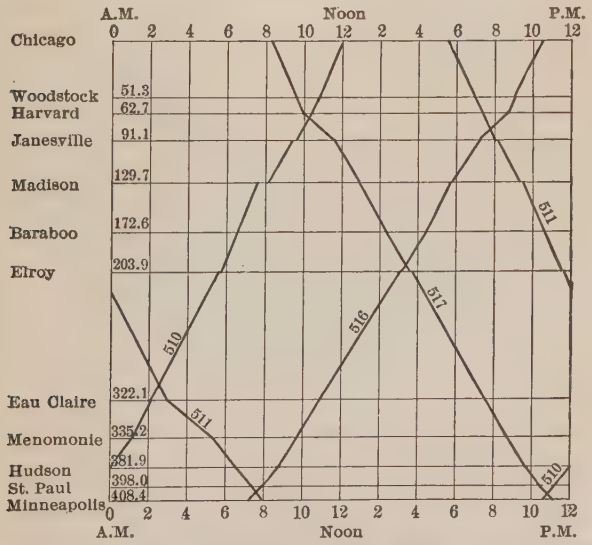


FIG. 20.—Graphical time table of certain passenger trains between Chicago and Minneapolis.

should always choose this ratio so that the graph will clearly convey its intended meaning.

Smooth curves are drawn through the plotted points of Fig. 18 (not straight lines as in Fig. 17), because in this case intermediate points have meaning; they represent temperatures at various times of the day.

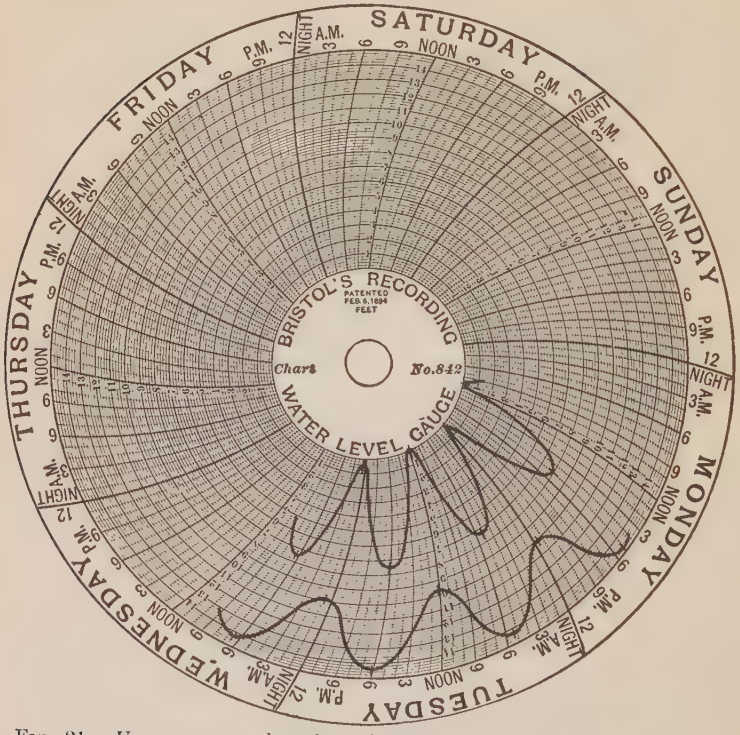


FIG. 21.—Upper curve, elevation of water in a well on Long Island. Lower curve, elevation of water in the nearby ocean.

Figure 19 is a barograph, or autographic record of the atmospheric pressure recorded during a balloon journey. Note that this graph makes use of neither rectangular nor polar coordinates. The scale of pressure is an *inverted* scale, increasing downward. The scale of time is an algebraic scale, the zero of which may be

arbitrarily selected at any convenient point. The scale of pressure is an arithmetical scale. The zero of the barometric scale corresponds to a perfect vacuum—no less pressure and hence, in this case, no negative value exists.

Figure 20 is a graphic time-table of certain passenger trains between Chicago and Minneapolis. The curves are not continuous, as in the barograph, but contain certain sudden jumps. What is the meaning of these? What indicates the speed of the trains? Where is the fastest track on this railroad? What shows the *meeting point* of trains?

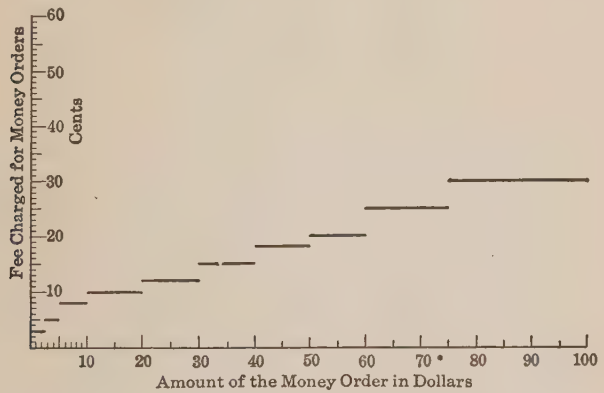


FIG. 22.—The graph of a discontinuous function.

If the diagram (Fig. 20) be wrapped around a vertical cylinder of such size that the two midnight lines coincide, then each train line may be traced through continuously from terminus to terminus. Functions having this remarkable property are said to be **periodic.** In the present case the trains run at the same time every day, that is periodically. In mathematical language, the position of the trains is said to be a **periodic function of the time.**

Figure 21 represents the fluctuation of the elevation of the ground water at a certain point near the seacoast on Long Island. The system of coordinates used in this graph is similar to the polar system.

The fluctuations are primarily due to the tidal wave in the nearby ocean. Is the curve periodic? What indicates the *rate of*

change in the elevation of the ground water? When is the elevation changing most rapidly? When is it changing most slowly?

Figure 22 represents the functional relation between the amount of a domestic money order and the fee. This is an excellent illustration of a *discontinuous function*. On account of the sudden jumps in the values of the fee, the fee is said to be a *discontinuous function* of the amount of the order.

14. Suggestions on the Construction of Graphs.—Two kinds of rectangular coordinate paper have been prepared for use with this book. Form $M1$ is ruled in centimeters and fifths. Form $M2$ is ruled without major divisions in uniform $\frac{1}{5}$-in. intervals.

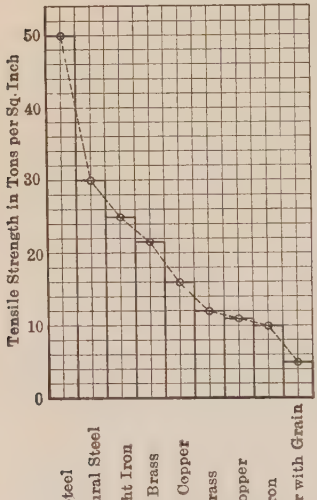


FIG. 23.—Graph showing tensile strength of certain structural materials.

It is a mistake to assume that more accurate work can be done on finely ruled than on more coarsely ruled squared paper. Quite the contrary is the case. Paper ruled to $\frac{1}{20}$-in. intervals does not permit interpolation within the small intervals, while paper ruled to $\frac{1}{10}$- or $\frac{1}{5}$-in. intervals permits accurate interpolation to one-tenth of the smallest interval. Form $M1$ is ruled to 2-mm. intervals, and is fine enough for any work. The centimeter unit has the very considerable advantage of permitting *twenty* of the units within the width of an ordinary sheet of letter paper ($8\frac{1}{2}$ by 11 in.), while *seven* is the largest number of inch units available on such paper.

In order to secure satisfactory results, the student must recognize that there are several varieties of statistical graphs, and that each sort requires appropriate treatment.

1. It is possible to make a useful graph when only *one* variable is given. Thus, Table II gives the ultimate tensile strength of various materials,

A graph showing these results is given in Fig. 23. There are two practical ways of showing the numerical values pertaining to each material, both of which are indicated in the diagram; either rectangles of appropriate height may be erected opposite the name of each material, or points marked by circles, dots, or crosses may be located at the appropriate height. It is obvious in this case that a smooth curve should *not* be drawn through these points —such a curve would be quite meaningless. In this case there are not two scales, but merely the single vertical scale. The horizontal axis bears merely the *names* of the different materials and has no *numerical* or *quantitative* significance. The result is, obviously, not the graph of a function, for there are not two

TABLE II.—ULTIMATE TENSILE STRENGTH OF VARIOUS MATERIALS

Material	Tensile strength, tons per square inch
Hard steel	50.0
Structural steel	30.0
Wrought iron	25.0
Drawn brass	21.5
Drawn copper	16.0
Cast brass	12.0
Cast copper	11.0
Cast iron	10.0
Timber, with grain	5.0

variables, but only one. The graph is merely a convenient expression for certain discrete and independent results arranged in order of descending magnitude.

2. It is possible to have a graph involving two variables in which it is either impossible or undesirable to represent the graph by a continuous curve or line. For example, Fig. 17 is a graph representing the maximum temperature on each day of a certain month. Because there is only one maximum temperature on each day, the value corresponding to this should be shown by an appropriate rectangle, or by marking a point by a circle, or by a

dot or cross, as in the preceding case. A continuous curve through these points has no meaning. The horizontal scale may be marked by the *names* of the days of the week or by *numbers*, but in either case the horizontal line is a true scale, as it corresponds to the lapse of the variable time. Sometimes, as in Fig. 17, graphs of this kind are represented by marking the appropriate points by dots or circles and then connecting the successive points by straight lines. These lines have no special meaning in such a case, but they aid the eye in following the succession of separate points.

If a graph be made of the noonday temperatures of each day of the same month referred to in Fig. 17, one of the same methods indicated above would be used to represent the results; that is, either rectangles, marked points, or marked points joined by lines. Although a smooth curve drawn through the known points would have a meaning (if correct), it is obvious that the noonday temperatures alone are not sufficient for determining its form. In all such cases a smooth curve should not be drawn.

3. If the data are reasonably sufficient, a smooth curve may, and often should, be drawn through the known points. Thus if the temperature be observed every hour of the day and the results be plotted, a smooth curve drawn carefully through the plotted points will probably very accurately represent the unknown temperatures at intermediate times. The same may safely be done in Exercises 3 and 4 below. In scientific work it is desirable to mark by circles or dots the values that are actually given, to distinguish them from the intermediate values "guessed" and represented by the smooth curve.

In addition to the above suggestions, the student should adhere to the following instructions:

4. Every graph should be marked with suitable numerals along both numerical scales.

5. Each scale of a statistical graph should bear in words a description of the magnitude represented and the name of the unit of measure used. These words should be printed in drafting letters and not written in script.

6. Each graph should bear a suitable title telling exactly what is represented by the diagram.

7. The selection of the units for the horizontal and vertical scales is an important practical matter in which common sense must control. It is obvious that in the third exercise given below 1 cm. = 1-ft. draft for the horizontal scale, and 1 cm. = 100 tons for the vertical scale will be units suitable for use on form $M1$.

Further instruction in practical graphing is given in Sec. **34.**

Exercises

1. Draw a statistical graph from the data given in the following table (see Fig. 17). Represent the plotted points by small distinct points, not by circles.

MAXIMUM AND MINIMUM TEMPERATURES AT MADISON, WISCONSIN, FOR OCTOBER, 1910

Date	Max. temp., ° F.	Min. temp., ° F.	Date	Max. temp., ° F.	Min. temp., ° F.
1	66	55	17	81	53
2	71	48	18	81	57
3	75	58	19	69	45
4	68	55	20	45	40
5	62	53	21	49	41
6	58	45	22	52	34
7	66	43	23	60	37
8	68	47	24	60	49
9	60	44	25	52	44
10	67	42	26	60	40
11	75	49	27	42	32
12	61	46	28	35	30
13	69	45	29	38	26
14	73	52	30	60	31
15	76	50	31	63	39
16	80	56			

2. Draw a statistical graph for the data given in the following table (see Fig. 18).

HOURLY AIR TEMPERATURES AT MADISON, WISCONSIN, MAY 14, 1910; OCTOBER 10, 1910

Time	May 14, 1910, temp., ° F.	Oct. 10, 1910, temp., ° F.	Time	May 14, 1910, temp., ° F.	Oct. 10, 1910, temp., ° F.
1 a. m.	37	44	1 p. m.	58	65
2 a. m.	35	44	2 p. m.	61	66
3 a. m.	35	44	3 p. m.	63	67
4 a. m.	34	43	4 p. m.	62	67
5 a. m.	34	43	5 p. m.	62	65
6 a. m.	35	42	6 p. m.	61	62
7 a. m.	37	43	7 p. m.	59	57
8 a. m.	42	45	8 p. m.	56	55
9 a. m.	46	51	9 p. m.	53	55
10 a. m.	49	55	10 p. m.	50	55
11 a. m.	54	60	11 p. m.	47	52
12 a. m.	56	62	12 p. m.	45	51

3. At the following drafts a ship has the displacements stated:

Draft in feet, h	15	12	9	6.3
Displacement in tons, T	2096	1512	1018	586

Plot on squared paper. What are the displacements when the drafts are 11 and 13 ft., respectively?

4. The following tests were made upon a steam-turbine generator:

Output in kilowatts, K...	1,190	995	745	498	247
Weight, pounds of steam consumed per hour, W.	23,120	20,040	16,630	12,560	8,320

Plot on squared paper. What are the probable values of K when W is 22,000 and also when W is 11,000?

5. The average temperature at Madison from records taken at 7 a.m. daily for 30 years is as follows:

Date	Degrees, Fahrenheit	Date	Degrees, Fahrenheit
Jan. 1,	14.0.	July 1,	67.5.
Feb. 1,	15.1.	Aug. 1,	64.0.
Mar. 1,	35.2.	Sept. 1,	55.4.
Apr. 1,	40.0.	Oct. 1,	44.1.
May 1,	53.9.	Nov. 1,	30.0.
June 1,	63.2.	Dec. 1,	18.3.

Make a suitable graph of these results on squared paper.

CHAPTER II

THE POWER FUNCTION

15. Definition of the Power Function.—Some of the most elementary and most important functions are those which consist of a single power of the variable, such as, for example, the functions x^2, x^3, $1/x$, $1/x^2$, $x^{\frac{2}{5}}$, etc. Such functions are all of the type x^n, where n is a positive or a negative constant, and are called **power functions.** It will be seen shortly that there is a close relationship between the function x^n and the function ax^n, and for this reason the name "power function" is also applied to the latter more general expression. In this chapter such functions will be studied, the usual procedure being to investigate first the simplest form x^n, and then to study the effect of introducing the factor a.

16. The Straight Line through the Origin.—The simplest case of the general power function ax^n occurs when $n = 1$. Special cases of the resulting function ax have already been observed. The equation $y = x$, for example, may be interpreted in rectangular coordinates as meaning that the abscissa of a point is always equal to its ordinate. The points of the plane which satisfy this condition lie, as we have seen, on the straight line bisecting the angles of the first and third quadrants. Again, the equation $y = 2x$ states that the ordinate is always twice the abscissa, and the locus of this equation is the straight line shown in Fig. 13. The functions $y = x$ and $y = 2x$ are special examples of the function $y = ax$, and it will now be shown that the graph of this equation is a straight line passing through the points (0, 0) and (1, a). In fact, since the pairs of numbers (0, 0) and (1, a) satisfy the equation, it is clear that the two points located by these two pairs of numbers are part of the locus of the equation. It only remains to show that any other point of the locus lies on the straight line through these two points. Let OD_2 be *any* value of x, and D_2P_2 the corresponding value of y, as determined from the

equation $y = ax$. Then (Fig. 24) the ratio $D_2P_2/OD_2 = a$. But the ratio $D_1P_1/OD_1 = a$. Therefore the triangles OD_1P_1 and OD_2P_2 are similar, angles D_1OP_1 and D_2OP_2 are equal, and P_2 (which is *any* point of the locus) lies on the straight line OP_1 extended.

The function $y = ax$ is the special case, when $n = 1$, of the power function ax^n. In this special case it is customary to use the letter m, rather than the letter a, for the coefficient of x. Using this notation, one would say that the graph of the equation $y = mx$ is a straight line through the origin and the point $(1, m)$.

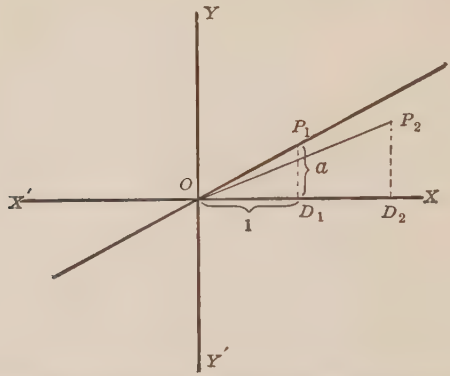


FIG. 24.—Diagram for use in proving that the graph of $y = ax$ is a straight line through the origin.

Exercises

1. Draw a line through the origin and the point (1, 3). What is its equation?

2. Draw a line through the origin and the point $(1, \frac{5}{2})$. What is its equation?

3. Draw a line through the origin and the point $(1, -1)$. What is its equation? Draw a line through the origin and the point $(1, -2)$. What is its equation?

4. Draw loci for the following: (*a*) the ordinate of any point of a certain locus is twice its abscissa; (*b*) the x of every point of a certain locus is half its y; (*c*) the y of a point is one-third of its x; (*d*) a point moves in such a way that its latitude is always treble its longitude; (*e*) the sum of the latitude and the longitude of a point is zero; (*f*) a

point moves so that the difference in its latitude and its longitude is always zero.

5. Graph the following equations: (*a*) $y = 4x$; (*b*) $y = \frac{1}{4}x$; (*c*) $3y = x$; (*d*) $2y = 3x$.

6. What is the equation of a straight line which is drawn through the origin and the point (3, 9)?

17. Slope.—The **slope** of a straight line is defined to be the **change** in y for an **increase** in x equal to 1. It will be represented in this book by the letter m.[1] In Fig. 25 the line A has the slope

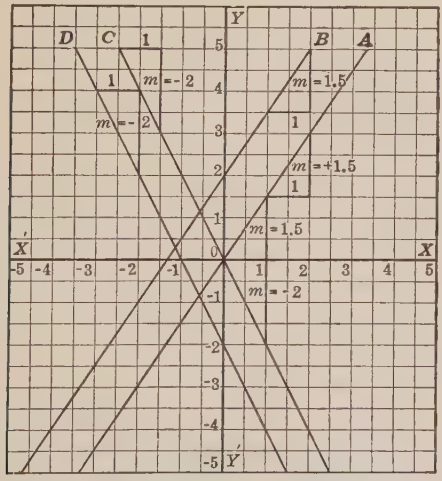


FIG. 25.—Lines of slope (1.5) and of slope (−2).

$m = 1.5$, for it is seen that at any point of the line the ordinate y gains 1.5 units for an increase of 1 in x. The line B, parallel to the line A, is also seen to have the slope equal to 1.5. The equation of the line A is, obviously, $y = 1.5x$. In the same figure the slope of the line C is -2, for at any point of this line the ordinate y *decreases* 2 units for an increase in x equal to 1. The equation of the line C is, obviously, $y = -2x$. Line D, parallel to line C, also has slope -2.

[1] This use of the letter m is consistent, as the student will see as he studies the section, with the use of m in the previous section.

If h be the change in y for an increase of x equal to k, then the slope m is the ratio h/k. Hence the practical method of determining the slope of a line drawn upon squared paper is: *Select two convenient points on the line rather far apart, and divide the change in y by the increase in x.*

The technical word *slope* differs from the word *slope* or *slant* in common language only in the fact that slope, in its technical use, is always expressed as the ratio of two algebraic numbers. In common language we speak of a "slope of 1 in 10," or a "grade of 50 ft. per mile," etc. In mathematics the equivalents are "slope = 1/10," "slope = 50/5280," etc.

As already indicated, the definition of slope requires us to speak in mathematics of **positive slope** and **negative slope.** A line of positive slope extends upward with respect to the standard direction OX and a line of negative slope extends downward with reference to OX.

In a similar way we may speak of the **slope of any curve** at a given point on the curve, meaning thereby the slope of the tangent line drawn to the curve at that point.

Exercises

1. Give the slopes of the lines in Exercises 1 to 6 of the preceding set of exercises.

2. Draw $y = x$; $y = 2x$; $y = 3x$; $y = 2x/3$; $y = x/2$; $y = x/4$; $y = -2x$; $y = -3x$; $y = 0x$.

18. Equation of a Straight Line. Intercepts.—In Fig. 26, the line MN expresses that the ordinate y for all points on the line, is, always 3 times the abscissa x; *i.e.* it says that $y = 3x$. The line HK is 2 units higher than MN, so that it states that "y is 2 more than $3x$." Thus the line HK has the equation $y = 3x + 2$. In Fig. 25 the line B is 2 units higher than the line $y = 1.5x$, hence its equation is $y = 1.5x + 2$. The line D is 2 units lower than the line C, whose equation is $y = -2x$, hence the equation of D is $y = -2x - 2$.

In general, since $y = mx$ is always a straight line, then $y = mx + b$ is a straight line, for the y of this locus is merely, in each case, the y of the former increased by the constant amount b (which may, of course, be positive or negative). Therefore, $y =$

$mx + b$ is a line parallel to $y = mx$. The line $y = mx + b$ is b units *higher* than, or above, the line $y = mx$ if b stands for a *positive* number and the line $y = mx + b$ is b units *lower* than, or *below* the line $y = mx$ if b stands for a *negative* number. The distance

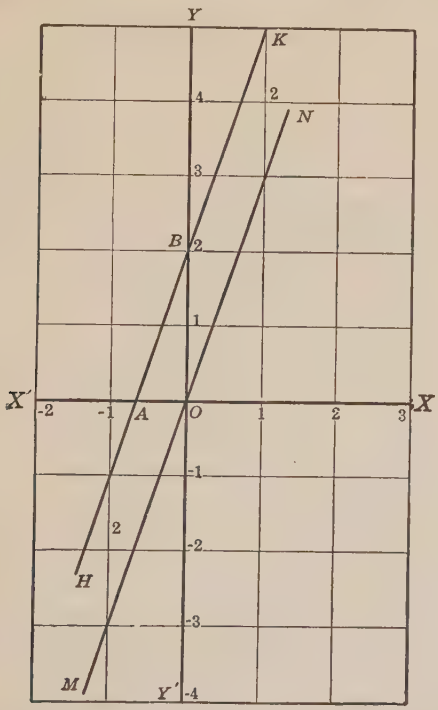


FIG. 26.—Intercepts. MN is the line $y = 3x$; HK is the line $y = 3x + 2$; OB, the Y-intercept of HK, is equal to 2; OA, the X-intercept, is equal to $-\frac{2}{3}$.

OB (Fig. 26) is equal to b, and is called the **Y-intercept** of the graph. The distance OA is equal to $-b/m$, for it is the value of x obtained from the equation when y is given the value zero. It is called the **X-intercept** of the locus.

The equation $\mathbf{y = mx + b}$ is called the **common equation of the straight line.**

Since the graph of the function $y = mx + b$ is, for any values of m and b, a straight line, this function is called the **linear function.** In Fig. 26 the line HK may be produced from the line MN by translating MN upward 2 units. On the other hand, the equation of the line HK namely, $y = 3x + 2$ or $y - 2 = 3x$, may be produced from the equation of the line MN, namely, $y = 3x$, by substituting $(y - 2)$ for y. That is, when $(y - 2)$ is substituted for y, the graph of the resulting equation can be obtained from the original graph by translating it upward 2 units. It is clear that this statement can be generalized. Since the line $y = mx + b$ or $y - b = mx$ is parallel to, but b units above, the line $y = mx$, we may say that the substitution of $(y - b)$ for y in the equation translates the graph b units in the positive y direction. This is the first illustration of a very important class of theorems which explain what changes take place in the *graph* of an equation when certain changes are made in the *equation*.

Illustration 1.—Sketch the line $y = 2x + 3$.

This line is 3 units higher than the line $y = 2x$. Hence through the point (0, 3) on the Y-axis, draw a line of slope 2, which is the required line.

Illustration 2.—Sketch the line $y = -2x - 1$.

The line is 1 unit lower than the line $y = -2x$. Hence through the point $(0, -1)$ on the Y-axis draw a line of slope -2. The line lies halfway between lines C and D (Fig. 25).

Illustration 3.—Draw the line whose equation is $4x - 2y - 3 = 0$.

Solve the equation for y by transposing the terms $4x$ and -3 to the right member and dividing both members by -2; then

$$y = 2x - \tfrac{3}{2}.$$

Hence through the point $(0, -3/2)$ on the Y-axis draw a line of slope 2.

Exercises

In connection with graphic work, the word "draw" is used in this text to indicate a careful construction, from which accurate values could be read. The word "sketch" will mean a less accurate, usually free-hand, sketch which is made without the use of extended numerical data, but which is nevertheless neat and clear enough to include therein the characteristic properties.

Exercises

1. Sketch, from inspection of the equations, the lines given by:

(*a*) $y = x$.
(*b*) $y = x + 1$.
(*c*) $y = x + 2$.
(*d*) $y = x + 3$.
(*e*) $y = x - 1$.
(*f*) $y = x - 2$.

2. Sketch, from inspection of the equations, the lines given by:

(*a*) $y = \frac{1}{2}x$.
(*b*) $y = \frac{3}{4}x$.
(*c*) $y = x$.
(*d*) $y = 2x$.
(*e*) $y = 3x$.
(*f*) $y = -\frac{1}{2}x$.
(*g*) $y = -x$.
(*h*) $y = -2x$.
(*i*) $y = -3x$.
(*j*) $y = \sqrt{2}x$.

3. Sketch the lines given by the equations:

(*a*) $x = 3$.
(*b*) $x = 5$.
(*c*) $x = -2$.
(*d*) $y = 1$.
(*e*) $y = 5$.
(*f*) $y = -3$.
(*g*) $y = 0$.
(*h*) $x = 0$.
(*i*) $x^2 = 4$.

4. Sketch and name the slope and the Y-intercept in each of the following:

(*a*) $y = x + 1$.
(*b*) $y = \frac{1}{2}x + 1$.
(*c*) $y = -2x + 4$.
(*d*) $y = 5x + 3$.
(*e*) $y = -5x - 2$.
(*f*) $y = 3x + 4$.
(*g*) $y = x - 6$.
(*h*) $y = \frac{2}{3}x + 8$.
(*i*) $y = -3x + 4$.
(*j*) $y = -\frac{1}{2}x - 3$.

5. Find the X-intercept and the Y-intercept for each of the following:

(*a*) $3x - 2y = 5$.
(*b*) $2x + y = 6$.
(*c*) $x - y = 7$.
(*d*) $y - 3x = 5$.
(*e*) $y - 2x - 6 = 0$.
(*f*) $2y + 3x + 5 = 0$.
(*g*) $x + y + 1 = 0$.
(*h*) $5y - 3x + 10 = 0$.

6. Name the slope and the Y-intercept in each of the following:

(*a*) $2y = x + 4$.
(*b*) $y - 2x - 3 = 0$.
(*c*) $y + \frac{2}{3}x + \frac{1}{3} = 0$.
(*d*) $2y + 3x = 4$.
(*e*) $2x - 3y = 6$.
(*f*) $4x = 3y - 6$.
(*g*) $10x - y = 7$.
(*h*) $ax - \frac{1}{2} = y$.
(*i*) $ax + by = c$.
(*j*) $\frac{x}{a} + \frac{y}{b} = 1$.

7. What is the equation of the X-axis? Of the Y-axis?

8. What is the equation of a line parallel to the X-axis 4 units above? 3 units above? 10 units below? 50 units below?

9. What is the equation of a line parallel to the Y-axis 3 units to the right? 20 units to the right? 7 units to the left? 100 units to the left?

10. Plot the following pairs of points on squared paper, and draw the line determined by each pair:

(a) $(-1, 3)$ and $(5, -6)$.
(b) $(-2, -5)$ and $(3, 4)$.
(c) $(1, 1)$ and $(7, -8)$.

Find the slope and, by means of similar triangles, find the Y-intercept of each line. Write the equation of each line by replacing m and b in $y = mx + b$ by the values found for slope and intercept. Test the correctness of the equations by substituting for x and y the coordinates of the given points.

11. A head of 100 ft. of water causes a pressure at the bottom of 43.4 lb. per square inch. Draw a graph showing the relation between head and pressure, for all heads of water from 0 to 200 ft.

SUGGESTION: There are several ways of proceeding. Let *pounds per square inch* be represented by abscissas or x, and *feet of water* be represented by ordinates or y. Since negative numbers are not involved in this exercise, the origin may be taken at or near the lower left corner of the squared paper. Draw a line through the points (0, 0) and (86.8, 200). This will be the required graph. Otherwise produce the equation $y = (100/43.4)\,x$ from the proportion $x : y = 43.4 : 100$ and then draw the graph from the fact that the latitude is always 100/43.4 of the longitude. In drawing this graph let 2 cm. on the X-axis represent 10 units, and 1 cm. on the Y-axis represent 10 units. Be sure that the scales are numbered and labeled in accordance with suggestions 4, 5, and 6 of Sec. **14**. On the X-axis mark only the points corresponding to hundreds of pounds, and on the Y-axis mark only the points corresponding to tens of feet.

12. From the straight line drawn in Exercise 11, find pressure measured in pounds corresponding to 13.1, 112.6, 93.7, and 187.5 ft. of water.

13. From the straight line drawn in Exercise 11, find the head in feet of water corresponding to 1123, 178, and 89 lb. per square inch.

14. A pressure of 1 lb. per square inch is equivalent to a column of 2.04 in. of mercury, or to one of 2.30 ft. of water. Draw a graph showing the relation between pressure expressed in feet of water and pressure expressed in inches of mercury.

SUGGESTION: Let x = inches of mercury and y = feet of water. First properly number and label the X-axis to express inches of mercury and number and label the Y-axis to express feet of water. Since negative numbers are not involved in this exercise, the origin may be taken at the lower left-hand corner of the squared paper. First locate

the point $x = 2.04$; $y = 2.30$ (which are the corresponding values given by the problem) and draw a line through it and the origin. This is the required locus, since at all points we must have the proportion $x:y = 2.04:2.30$, which says that the ordinate of every point of the locus is 2.30/2.04 times the abscissa of that point.

15. Cast-iron pipe, class *A* (manufactured for heads under 100 ft.), weighs, per foot of length: 4 in., 20.0 lb.; 6 in., 30.8 lb.; 8 in., 42.9 lb. Upon a single sheet of squared paper, construct a graph for each size of pipe, showing the cost per foot for all variations in market price between \$20 and \$40 per ton.

SUGGESTION: If the horizontal scale be selected to represent *price per ton*, the scale may begin at 20 and end at 40, as this covers the range required by the problem. Therefore, let 1 cm. represent \$1. Since the range of prices is from 1 to 2 cts. per pound, the cost per foot will range from 20 to 40 cts. for 4-in. pipe and from 42.9 to 85.8 cts. for 8-in. pipe. Hence for the vertical scale 10 cts. may be represented by 2 cm. In this case the vertical scale may quite as well begin at 0 cts. instead of at 20 cts. as there is plenty of room on the paper.

16. Show that the shortest distance between $y = mx$ and $y = mx + b$ is not b, but $b/\sqrt{1 + m^2}$.

19. Line with Slope and One Point, or with Two Points Given.— The equation of any line parallel to the Y-axis is of the form $x = a$, which is an equation in which the variable y does not appear. The equation of all other lines may be written in the form

$$y = mx + b, \tag{1}$$

in which m is the slope of the line and b is the Y-intercept. Two important special cases are explained below:

Illustration 1.—Find the equation of the line of slope 4 which passes through the point (2, 3).

Since $m = 4$, Eq. (1) becomes

$$y = 4x + b. \tag{2}$$

Replacing x by 2 and y by 3, we get

$$3 = 8 + b, \text{ or } b = -5.$$

Hence the equation of the line of slope 4 passing through (2, 3) is

$$y = 4x - 5. \tag{3}$$

Illustration 2.—Find the equation of the line passing through the points (2, 3) and (4, 1).

Substituting the given values of x and y in Eq. (1) we have

$$3 = 2m + b$$
$$1 = 4m + b.$$

Solving these equations for the *two* unknown numbers, m and b, we find

$$b = 5$$
$$m = -1,$$

so that the equation of the line passing through the given points is

$$y = -x + 5.$$

In like manner, the equation of a line passing through any two given points may be found. In geometry we learned that two points determine a straight line, and in the present problem the coordinates of two given points are sufficient for the determination of the equation of the line.

Exercises

1. Find the equation of the line determined by each of the following conditions:

(*a*) Passes through (2, 5) and has slope 3.

(*b*) Passing through $(-2, 6)$ and has slope $-2\frac{1}{2}$.

(*c*) Passes through $(4, -1)$ and has slope 7.41.

2. Find the equation of the line determined by each of the following pairs of points:

(*a*) (3, 2) and (1, 5).

(*b*) (1, 2) and $(-2, 6)$.

(*c*) (4, 6) and $(3, -2)$.

(*d*) (0, 0) and $(-2, -3)$.

3. Show that the equation of the line passing through $(a, 0)$ and $(0, b)$ may be written in the form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

4. Find the equations of the three sides of the triangle whose vertices are the points (0, 3), (2, 4) and (5, 9).

In each of the following exercises certain observed data are tabulated, which will be found in each case to give points lying on a straight line. The law connecting y and x must then be of the form $y = mx + b$.

5. Find the law connecting y and x when the following corresponding values are given:

x	10	25	54	72
y	17	47	105	141

Hint: In plotting, take the origin at the lower left corner of the squared paper. Let 2 cm. represent 10 units on the X-axis and 1 cm. represent 10 units on the Y-axis. To find the slope divide the change in y, or $(141 - 17)$, by the increase in x, or $(72 - 10)$, which gives 2. Find the Y-intercept by method of Illustration 1.

6. Find the law connecting x and y from the following data:

x	12.0	15.3	17.8	19.0
y	24.2	29.0	32.6	34.2

Hint: Take origin near left lower corner of the squared paper and let 4 cm. equal 10 units on each axis.

7. L is the latent heat of steam at a temperature t° C. Find a simple formula, giving L in terms of t.

t	75	90	100	115	125
L	554	544	536	526	519

Hint: Call the lower left corner of the squared paper the point $t = 75$, $L = 500$. Let 1 cm. equal 5 units on each axis.

8. V is the volume in cubic centimeters of a certain weight of gas at temperature t°C., the pressure being constant. Find the law connecting V and t.

t	27.0	33.0	40.0	55.0	68.0
V	109.9	112.0	114.7	120.1	125

Hint: Call the lower left corner of the squared paper the point $t = 25$, $V = 100$. Let 2 cm. equal 5 units on the t scale and 1 cm. equal 1 unit on the V scale.

20. The Intersections of Straight Lines.—If a point lies on a line, the coordinates of the point satisfy the equation of the line. The intersection of two straight lines is a point which lies on each line, and whose coordinates, therefore, satisfy the equation of each line. Thus it is possible to solve a pair of simultaneous linear equations in two unknowns by constructing the graphs of these two equations, and noting the coordinates of their point of intersection. Conversely, the geometric problem of finding the intersection of two straight lines is equivalent to the algebraic problem of solving simultaneously the equations of the two lines.

Exercises

1. Obtain graphically the solution of the pair of simultaneous equations: $4y - x = 2$, $y + x = 3$.

2. Solve graphically:

$$(a)\begin{cases} y = 2x + 3. \\ y = x - 4. \end{cases} \qquad (b)\begin{cases} 4x - 3y = 2. \\ x + 3y = 3. \end{cases}$$

3. Find the coordinates of the point of intersections of the following pairs of lines:

$$(a)\begin{cases} 4x - y = 3. \\ x + 4y = 2. \end{cases} \qquad (b)\begin{cases} x + y = 10. \\ x - 2y = 5. \end{cases}$$

$$(c)\begin{cases} x + 7y = 13. \\ x - y = 4. \end{cases} \qquad (d)\begin{cases} 2x - 3y - 5 = 0. \\ 3x + 7y + 13 = 0. \end{cases}$$

4. Interpret geometrically the difficulty that arises when one attempts to solve as simultaneous equations the pairs:

$$(a)\begin{cases} x + y = 7. \\ 2x + 2y = 14. \end{cases} \qquad (b)\begin{cases} y - 3x = 2. \\ y - 1 = 3x. \end{cases}$$

21. The Graph of x^2.—Next to the linear function in simplicity is the function x^2, or, in more general form, ax^2. This function arises in many cases of practical importance. Thus the area of a circle depends upon the *square* of the radius; the distance traversed by a falling body depends upon the square of the elapsed time; the pressure upon a flat surface exposed directly to the wind depends upon the square of the velocity of the wind; the heat generated in an electric current in a given time depends upon the square of the number of amperes of current, etc. Each of these relations is expressed by an equation of the form $y = ax^2$, in which x stands for the number of units in one of the variable quantities (radius of the circle, time of fall, velocity of the wind, amperes of current, respectively, in the above-named cases) and in which y stands for the other variable dependent upon these. The number a is a constant which has a value suitable to each particular problem, but, in general, is not the same constant in different problems. Thus, if y be taken as the area of a circle, $y = \pi x^2$, in which x is the radius measured in feet or inches, etc., and y is measured in square feet or square inches, etc.; or if s be the distance in feet traversed by a falling body, then $s = 16.1t^2$, where t stands for the

elapsed time in seconds. In one case the value of the constant a is 3.1416 and in the other its value is 16.1.

Let us first graph the abstract law or equation $y = x^2$, in which a concrete meaning is not assumed for the variables x and y but in which both are thought of as abstract variables. First form a suitable table of values for x and x^2 as follows:

x	x^2 or y	x	x^2 or y
− 3.0	9.00	0.2	0.04
− 2.5	6.25	0.4	0.16
− 2.0	4.00	0.6	0.36
− 1.8	3.24	0.8	0.64
− 1.6	2.56	1.0	1.00
− 1.4	1.96	1.2	1.44
− 1.2	1.44	1.4	1.96
− 1.0	1.00	1.6	2.56
− 0.8	0.64	1.8	3.24
− 0.6	0.36	2.0	4.00
− 0.4	0.16	2.5	6.25
− 0.2	0.04	3.0	9.00
0.0	0.00		

Here we have a series of pairs of values of x and y which are associated by the relation $y = x^2$. Using the x of each pair of values as abscissa with its corresponding y, there can be located as many points as there are pairs of values in the table, and the array of points thus marked may be connected by a freely drawn curve. To draw the curve upon coordinate paper, (form $M1$), the origin may be taken near the lower midpoint of the sheet, and 2 cm. used as the unit of measure for x and y. If the points given by the pairs of values are not located fairly close together, it is obvious that a smooth curve cannot be satisfactorily sketched between the points until intermediate points are located by using intermediate values of x in forming the table of values. The student should think of the curve as extending indefinitely beyond the limits of the sheet of paper used; the entire locus consists of the part actually drawn and of the endless portions that must be followed in imagination beyond the range of the paper. If the graph of $y = x^2$ be folded about the Y-axis OY, it will be noted at once that the left and right portions of the

curve will exactly coincide. The student will explain the reason for this fact.

22. Parabolic Curves.—The equations $y = x$, $y = x^2$, $y = x^{3/2}$, $y = x^3$ should be graphed by the student on a single sheet of coordinate paper, using 2 cm. as the unit of measure in each case.[1]

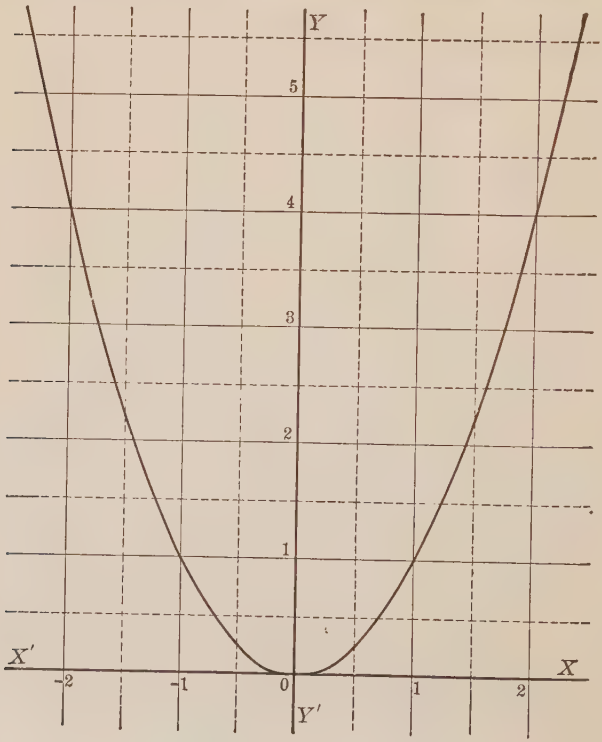


FIG. 27.—The parabola $y = x^2$.

Table III may be used to save numerical computation in the construction of the graphs of these power functions. As in the case of $y = x^2$, a smooth curve should be sketched free-hand through the points located by means of the table of values, and

[1] When squared paper ruled in inches is used instead of form *M*1, 1 in. should be taken as the unit of measure.

intermediate values of x and y should be computed when doubt exists in the mind of the student concerning the course of the curve between any two points.

TABLE III

x	x^2	x^3	$\sqrt{x}$	$\sqrt[3]{x}$	$x^{3/2}$	$1/x$	$1/x^2$
0.2	0.04	0.008	0.447	0.585	0.089	5.000	25.000
0.4	0.16	0.064	0.632	0.737	0.252	2.500	6.250
0.6	0.36	0.216	0.775	0.843	0.465	1.667	2.778
0.8	0.64	0.512	0.894	0.928	0.715	1.250	1.563
1.0	1.00	1.000	1.000	1.000	1.000	1.000	1.000
1.2	1.44	1.728	1.095	1.063	1.312	0.8333	0.6944
1.4	1.96	2.744	1.183	1.119	1.657	0.7143	0.5102
1.6	2.56	4.096	1.265	1.170	2.034	0.6250	0.3906
1.8	3.24	5.832	1.342	1.216	2.415	0.5556	0.3086
2.0	4.00	8.000	1.414	1.260	2.828	0.5000	0.2500
2.2	4.84	10.65	1.483	1.301	3.263	0.4545	0.2066
2.4	5.76	13.82	1.549	1.339	3.717	0.4167	0.1736
2.6	6.76	17.58	1.612	1.375	4.193	0.3846	0.1479
2.8	7.84	21.95	1.673	1.409	4.685	0.3571	0.1276
3.0	9.00	27.00	1.732	1.442	5.196	0.3333	0.1111
3.2	10.24	32.77	1.789	1.474	5.724	0.3125	0.0977
3.4	11.56	39.30	1.844	1.504	6.269	0.2941	0.0865
3.6	12.96	46.66	1.897	1.533	6.831	0.2778	0.0772
3.8	14.44	54.87	1.949	1.560	7.407	0.2632	0.0693
4.0	16.00	64.00	2.000	1.587	8.000	0.2500	0.0625
4.2	17.64	74.09	2.049	1.613	8.608	0.2381	0.0567
4.4	19.36	85.18	2.098	1.639	9.229	0.2273	0.0517
4.6	21.16	97.34	2.145	1.663	9.866	0.2174	0.0473
4.8	23.04	110.6	2.191	1.687	10.42	0.2083	0.0434
5.0	25.00	125.0	2.236	1.710	11.18	0.2000	0.0400
5.2	27.04	140.6	2.280	1.732	11.85	0.1923	0.0370
5.4	29.16	157.5	2.324	1.754	12.66	0.1852	0.0343
5.6	31.36	175.6	2.366	1.776	13.25	0.1786	0.0319
5.8	33.64	195.1	2.408	1.797	13.97	0.1724	0.0297
6.0	36.00	216.0	2.449	1.817	14.70	0.1667	0.0278
6.2	38.44	238.3	2.490	1.837	15.44	0.1613	0.0260
6.4	40.96	262.1	2.530	1.857	16.19	0.1563	0.0244
6.6	43.56	287.5	2.569	1.876	16.96	0.1515	0.0230
6.8	46.24	314.4	2.608	1.895	17.33	0.1471	0.0216
7.0	49.00	343.0	2.646	1.913	18.52	0.1429	0.0204
7.2	51.84	373.2	2.683	1.931	19.32	0.1389	0.0193
7.4	54.76	405.2	2.720	1.949	20.13	0.1351	0.0183
7.6	57.76	439.0	2.757	1.966	20.95	0.1316	0.0173
7.8	60.84	474.6	2.793	1.983	21.79	0.1282	0.0164
8.0	64.00	512.0	2.828	2.000	22.63	0.1250	0.0156
8.2	67.24	551.4	2.864	2.017	23.48	0.1220	0.0149
8.4	70.56	592.7	2.898	2.033	24.35	0.1190	0.0142
8.6	73.96	636.1	2.933	2.049	25.22	0.1163	0.0135
8.8	77.44	681.5	2.966	2.065	26.11	0.1136	0.0129
9.0	81.00	729.0	3.000	2.080	27.00	0.1111	0.0123
9.2	84.64	778.7	3.033	2.095	27.91	0.1087	0.0118
9.4	88.36	830.6	3.066	2.110	28.82	0.1064	0.0113
9.6	92.16	884.7	3.098	2.125	29.74	0.1042	0.0109
9.8	96.04	941.2	3.130	2.140	30.68	0.1020	0.0104
10.0	100.00	1000.0	3.162	2.154	31.62	0.1000	0.0100

All of the graphs here considered have one important property in common, namely, they all pass through the points (0, 0) and

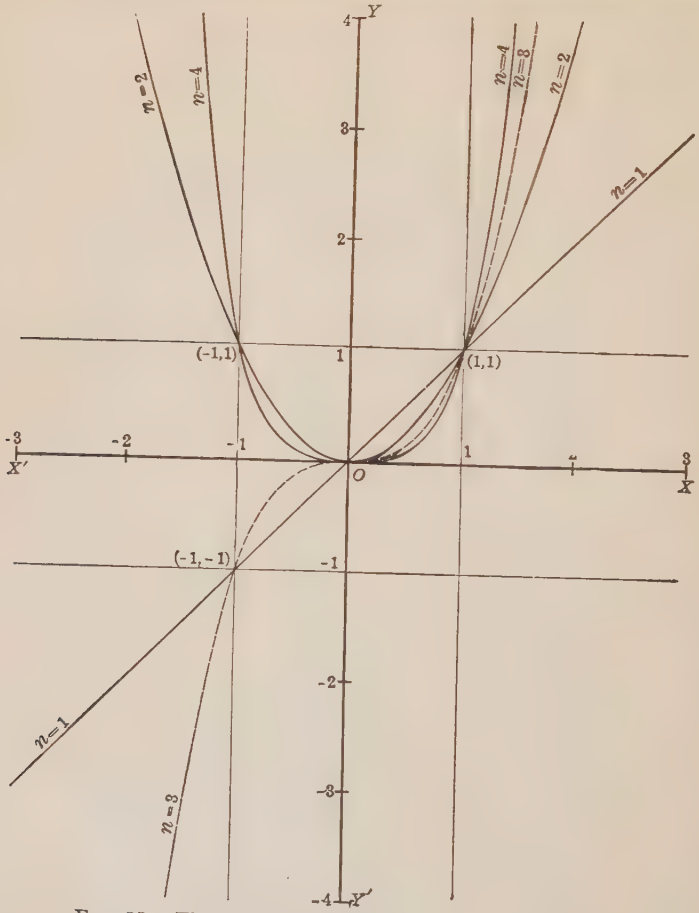


FIG. 28.—The parabolas $y = x^n$ for $n = 1, 2, 3$, and 4.

(1, 1). It is obvious that this property may be affirmed of any curve of the class $y = x^n$, *if n be a positive number*. These curves are known collectively as *curves of the parabolic family*, or simply

parabolic curves. The curve $y = x^2$ is called **the parabola.** $y = x^3$ is called the **cubical parabola.** $y = x^{3/2}$ is called the **semi-cubical parabola,** etc. Curves for negative values of n do not pass through the point (0, 0) and are otherwise quite distinct. They are known as curves of the **hyperbolic type,** and will be discussed later.

The student should cut patterns of the parabola, the cubical parabola, and the semi-cubical parabola out of heavy paper for use in drawing these curves when required. Each pattern should have drawn upon it either the X- or the Y-axis and one of the unit lines, to assist in properly adjusting the pattern upon squared paper.

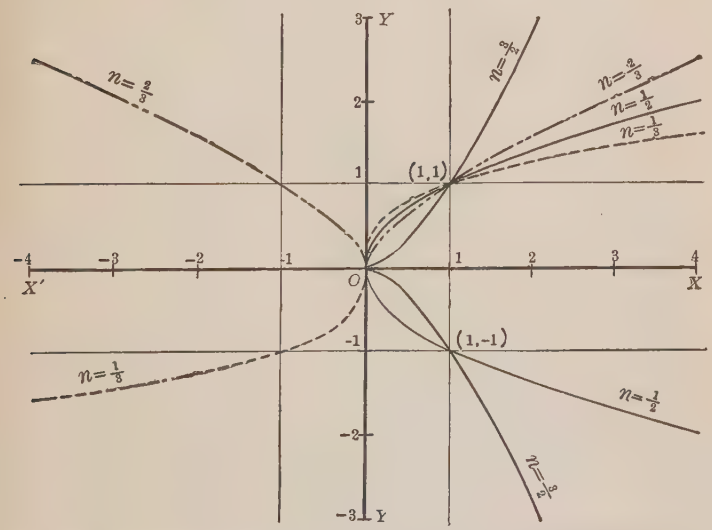


FIG. 29.—The parabolas $y = x^n$ for $n = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}$. Especially note the graphs of $y = x^{3/2}$ and $y = x^{2/3}$.

23. Symmetry.—In geometry a distinction is made between two kinds of symmetry of plane figures—symmetry with respect to a *line* and symmetry with respect to a *point.* A plane figure is symmetrical with respect to a given line if the two parts of the figure exactly coincide when folded about that line. Thus the let-

ters M and W are each symmetrical with respect to a vertical line drawn through the vertex of the middle angles. We have already noted that $y = x^2$ is symmetrical with respect to OY.

A plane figure is symmetrical with respect to a given point when the figure remains unchanged if rotated 180° in its own plane about an axis perpendicular to the plane at the given point. Thus, the letters N and Z are each symmetrical with respect to the midpoint

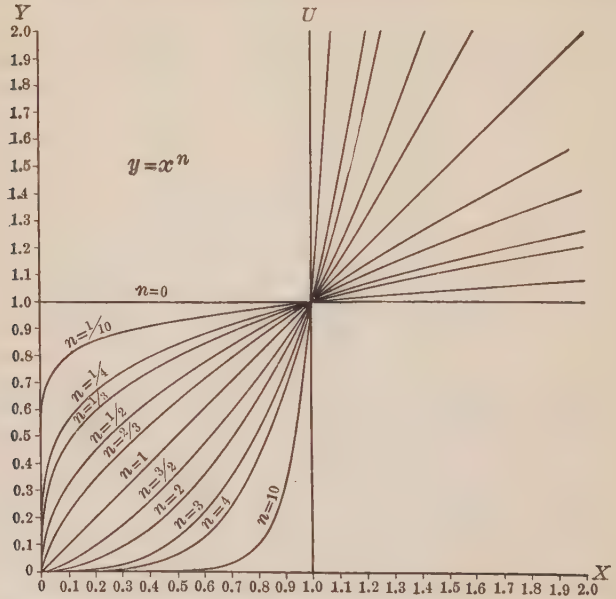


FIG. 30.—Graph of the power function for $n > 0$ (parabolic curves) in the first quadrant.

of their central line. The letters H and O are symmetrical both with respect to lines and with respect to a point. Which sort of symmetry is possessed by the curve $y = x^3$? Why?

Another definition of symmetry with respect to a point is perhaps clearer than the one given by the above statement: A curve is said to be symmetrical with respect to a given point O when all lines drawn through the given point and terminated by the curve are bisected at the point O.

What kind of symmetry with respect to one of the coordinate axes or to the origin (as the case may be) does the point (2, 3) bear to the point (−2, 3)? To the point (−2, −3)? To the point (2, −3)?

Note that symmetry of the first kind means that a plane figure is unchanged when turned 180° about *a certain line in its plane*, and that symmetry of the second kind means that a figure is unchanged when turned 180° about a certain line *perpendicular to its plane*.

The curves in Figs. 27 to 30 are sketched from a limited number of points only, but any number of additional values of x and y may be tabulated and the accuracy, as well as the extent, of the graph be made as great as desired. A number of graphs of power functions are shown as they appear in the *first quadrant* in Figs. 30 and 34. The student should explain how to draw the portions of the curves lying in the other quadrants from the part appearing in the first quadrant.

Exercises

1. On coordinate paper draw the curves $y = x^2$, $y = x^3$, $y = x^{3/2}$, $y = x^5$, using 2 cm. as the unit of measure. On the same sheet draw the lines $x = \pm 1$, $y = \pm 1$, $y = \pm x$.

2. On coordinate paper sketch the curves $x = y^2$, $x = y^3$, $x = y^{3/2}$, $x = y^5$. Compare with the curves of Exercise 1.

3. Sketch and discuss the curves $y = x^{1/2}$, $y = x^{1/3}$, $y = x^{1/4}$. Can any of these curves be drawn from patterns made from the curves of Exercise 1? Why?

4. Draw the curve $y^2 = x^4$. Compare with the curve $y = x^2$.

5. Name in each case the quadrants in which the curves of Exercises 1 to 4 lie, and state the reasons why each curve exists in certain quadrants and not in the other quadrants.

24. Discussion of the Parabolic Curves.—Draw the straight lines $x = 1$, $x = -1$, $y = 1$, $y = -1$ upon the same sheet upon which a number of parabolic curves have been drawn. These lines together with the coordinate axes divide the plane into a number of rectangular spaces. In Fig. 31 these spaces are shown divided into two sets, those represented by the cross-hatching, and those shown plain. *The cross-hatched rectangular spaces contain the lines* $y = x$ *and* $y = -x$ *and also all curves of the parabolic*

type. No parabolic curve enters the rectangular strips shown plain in Fig. 31.

The line $y = x$ divides the spaces occupied by the parabolic curves into equal portions. Why does the curve $y = x^2$ (in the first quadrant) lie below this line in the interval $x = 0$ to $x = 1$, but above it in the interval to the right of $x = 1$? On the other hand, why does the curve $y = x^{1/2}$, or $y^2 = x$ (in the first quadrant), lie above the line $y = x$ in the interval $x = 0$ to $x = 1$ and below $y = x$ in the interval to the right of $x = 1$?

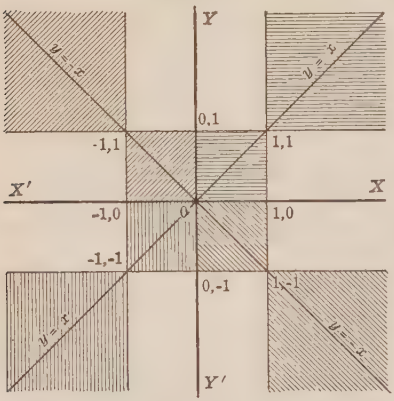


FIG. 31.—The regions of the parabolic and the hyperbolic curves. All parabolic curves lie within the cross-hatched region. All hyperbolic curves lie within the region shown plain.

One part of the parabolic curve $y = x^n$ always lies in the first quadrant. If n be an even integer, another part of the curve lies in which quadrant? If n be an odd integer, the curve lies in which quadrants?

If the exponent n of any power function be a positive fraction, say m/r, the equation of the curve may be written $y^r = x^m$. If in this case both m and r be odd, the curve lies in which quadrants? If m be even and r be odd, the curve lies in which quadrants? If m be odd and r be even, the curve lies in which quadrants? If both m and r be even, the curve lies in which quadrants?

A curve which is *symmetrical to another curve with respect to a line* may, figuratively, be spoken of as the **reflection** or **image** of the second curve in a mirror represented by the given line.

Exercises

Exercises 1 to 4 refer to curves in the first quadrant only.

1. The expressions x^2, $x^{3/2}$, x^3, x^4 are numerically less than x for values of x between 0 and 1. How is this fact shown in Fig. 30?

2. The expressions x^2, $x^{3/2}$, x^3, x^4 are numerically greater than x for all values of x numerically greater than unity. How is this fact pictured in Fig. 30?

3. For values of x between 0 and 1, $x^4 < x^3 < x^2 < x^{3/2} < x$. For values $x > 1$, $x^4 > x^3 > x^2 > x^{3/2} > x$. Explain how each of these facts is expressed by the curves of Fig. 30.

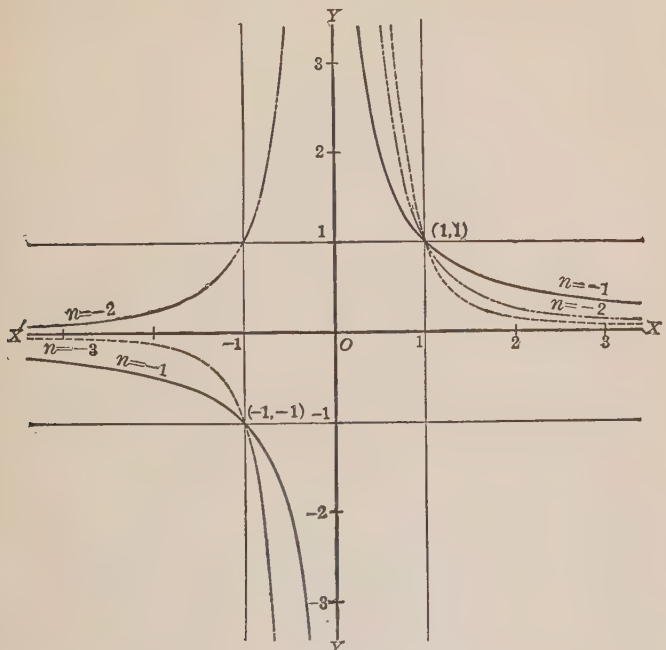


FIG. 32.—The hyperbolas $y = x^n$ for $n = -1$, -2, and -3.

4. Sketch on a single sheet of squared paper, without tabulating the numerical values, the following loci: $y = x^{10}$, $y = x^{0.1}$, $y = x^{100}$, $y = x^{0.01}$.

The following are to be discussed for all quadrants:

5. Sketch, without tabulating numerical values, the following loci: $y^2 = x^4$, $y^4 = x^6$, $y^4 = x^2$, $y^3 = x^5$, $y^5 = x^3$.

6. Show that $y = -x^2$ is the reflection of $y = x^2$ in the X-axis.

7. Sketch the following: $y = -x$, $y = -x^2$, $y = -x^3$, $y^3 = -x^2$, $y^2 = -x^3$.

8. A ball rolls down a smooth inclined plane making an angle of 45° with the horizontal. The distance s measured in feet along the incline is given by the formula

$$s = 11.4t^2,$$

where t is time in seconds. Draw a graph for this equation. Let time t be represented by distances along the axis of abscissas and dis-

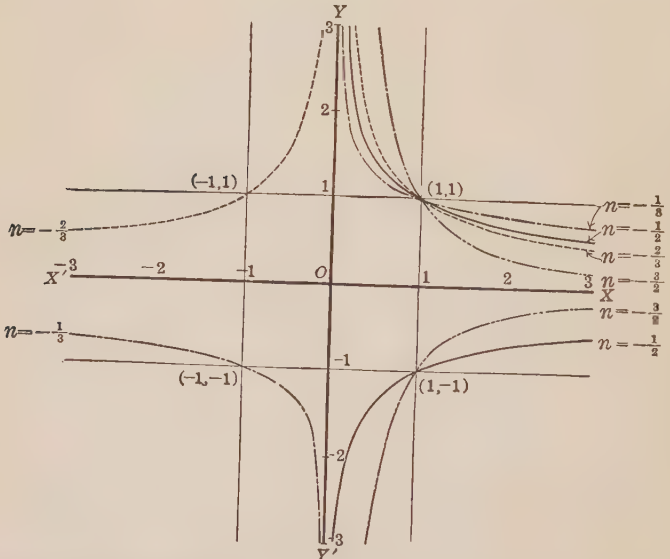


FIG. 33.—The hyperbolas $y = x^n$ for $n = -3/2$, $-2/3$, $-1/2$, and $-1/3$.

tance s along the axis of ordinates. Let 2 cm. represent 1 sec. on one axis and 10 ft. on the other.

25. Hyperbolic Curves.—Loci of equations of the form $yx^n = 1$, or $y = 1/x^n$, where n is positive, are called **hyperbolic curves.** The fundamental curve $xy = 1$, or $y = 1/x$ is called the **rectangular hyperbola.** Its graph is given in Fig. 32, but the curve should be drawn independently by the student, using 2 cm. as

the unit of measure. Its relation to the X- and the Y-axes is most characteristic. For a very small positive value of x, the value of y is very large, and, as x approaches 0, y increases indefinitely. But the function is not defined for $x = 0$, for the product xy cannot equal 1 if x be zero. For numerically small but negative

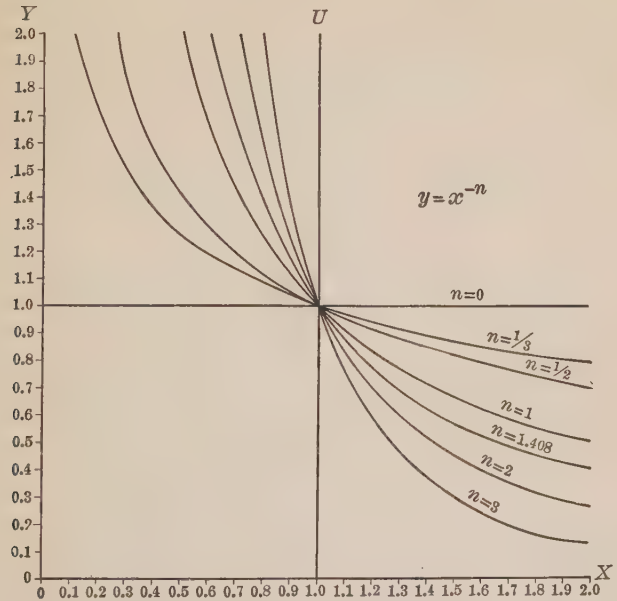


FIG. 34.—Hyperbolic curves in the first quadrant. $y = 1/x^{1.408}$ is the adiabatic curve for air.

values of x, y is negative and numerically very large, and becomes numerically larger as x approaches 0.

Instead of saying that "y increases in value without limit," it is just as common to say "y becomes infinite"; in fact, "infinite" is merely the Latin equivalent of "no limit." It is often written $y = \infty$. This is a mere abbreviation for the longer expressions, "y becomes infinite" or "y increases in value without limit." The student must be cautioned that the symbol ∞ does not stand

for a *number*, and that "$y = \infty$" must not be interpreted in the same way that "$y = 5$" is interpreted.

As x increases from numerically large negative values to 0, y continually decreases and becomes negatively infinite (abbreviated $y = -\infty$). As x decreases from numerically large positive values to 0, y continually increases and becomes infinite. Thus, in the neighborhood of $x = 0$, y is **discontinuous,** and, in this case, the discontinuity is called an **infinite discontinuity.**

On account of the symmetry in $xy = 1$, if we look upon x as a function of y, all of the above statements may be repeated, merely interchanging x and y wherever they occur. Thus, there is an infinite discontinuity in x, as y passes through the value 0.

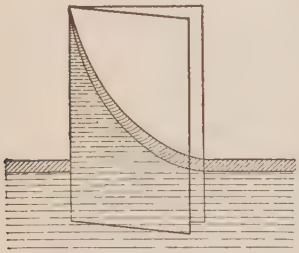


FIG. 35.—A hyperbola formed by capillary action of two converging plane plates.

The lines XX' and YY' which these curves approach as near as we please, but never meet, are called the **asymptotes** of the hyperbola.

All other curves of the hyperbolic family, such as $yx^2 = 1$, $xy^2 = 1$, $y^3x^2 = 1$, $y^4x^4 = 1$, and the like, approach the X- and the Y-axes as asymptotes. The rates at which they approach the axes depend upon the relative magnitudes of the exponents of the powers of x and y; the quadrants in which the branches lie depend upon the oddness or the evenness of these exponents.

Exercises

1. Draw accurately upon squared paper the loci, $xy = 1$, $xy^2 = 1$, $x^2y = 1$. Use 2 cm. as unit and make a pattern for $xy = 1$.

2. Show that the curves of the hyperbolic type lie in the rectangular regions shown plain, or not cross-hatched, in Fig. 31.

3. In what quadrants do the branches of $x^{15}y^7 = 1$ lie?

4. How does the locus of $x^2y^2 = 1$ differ from that of $xy = 1$?

5. Sketch, showing the essential character of each locus, the curves $x^3y = 1$, $x^{10}y = 1$, $x^{1000}y = 1$.

6. Show that $xy = a$ passes through the point $(\sqrt{a}, \sqrt{a})$; that $xy = a^2$ passes through (a, a) and can be made from $xy = 1$ by "stretching" (if $a > 1$) both abscissas and ordinates of $xy = 1$ in the ratio $1 : a$.[1]

26. Symmetry.—Some of the facts of symmetry respecting two portions of the same parabola or hyperbola may be readily extended by the student to other curves. First answer the following questions:

How are the points (a, b) and $(-a, b)$ related to the Y-axis?

How are the points (a, b) and $(a, -b)$ related to the X-axis?

How are the points (a, b) and (b, a) related to the line $y = x$? Prove the result by plane geometry.

The following may then be readily proved by the student:

Theorems on Loci

I. *If x be replaced by $(-x)$ in any equation containing x and y, the new graph is the reflection of the former in the axis YY'.*

II. *If y be replaced by $(-y)$ in any equation containing x and y, the new graph is the reflection of the former in the axis XX'.*

III. *If x and y be interchanged in any equation containing x and y, the new graph is the reflection of the former in the line $y = x$.*

By means of these three theorems, the student will prove the following statements:

1. If an equation remains unchanged when x is replaced by $-x$, its graph is symmetrical with respect to the *Y-axis.*

2. If an equation remains unchanged when y is replaced by $-y$, its graph is symmetrical with respect to the X-axis.

3. If an equation remains unchanged when x and y are interchanged, its graph is symmetrical with respect to the line $y = x$.

4. If an equation remains unchanged when x is replaced by $-x$ and y by $-y$, its graph is symmetrical with respect to the origin.

27. The Variation of the Power Function.—The symmetry of the graphs of the power function with respect to certain lines and points, while of interest geometrically, nevertheless does not constitute the most important fact in connection with these functions.

[1] To "elongate" or "stretch" in the ratio 2:3 means to change the length of a line segment so that (original length): (new or stretched length) = 2:3.

Of more importance is *the law of change of value* or *the law by which the function varies.* Thus returning to a table of values for the power function x^2 for the first quadrant,

x	x^2	d
0	0	
		$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	
		$\frac{3}{4}$
1	$\frac{4}{4}$	
		$\frac{5}{4}$
$\frac{3}{2}$	$\frac{9}{4}$	
		$\frac{7}{4}$
2	$\frac{16}{4}$	
		$\frac{9}{4}$
$\frac{5}{2}$	$\frac{25}{4}$	
		$\frac{11}{4}$
3	$\frac{36}{4}$	
		$\frac{13}{4}$
$\frac{7}{2}$	$\frac{49}{4}$	
		$\frac{15}{4}$
4	$\frac{64}{4}$	

we note that as x changes from 0 to $\frac{1}{2}$ the function grows by the small amount $\frac{1}{4}$. As x changes from $\frac{1}{2}$ by another increment of $\frac{1}{2}$ to the value 1, the function increases by $\frac{3}{4}$ to the value 1. As x grows by successive steps or increments of $\frac{1}{2}$ unit each, it is seen that x^2 grows by increasingly greater and greater steps, until finally the change in x^2 produced by a small change in x becomes very large. Thus, the step-by-step increase in the function is a rapidly augmenting one, as is shown by the column of differences headed "d" in the table. Even more rapidly does the function x^3 gain in value as x grows in value. On the contrary, for positive values of x the power functions $1/x$, $1/x^2$, $1/x^3$, etc. *decrease* in value as x grows in value. Referring to the definition of the slope of a curve given in Sec. **17,** we see that the parabolic curves have a positive slope in the first quadrant, while the hyperbolic curves have always a negative slope in the first quadrant.

The law of the power function is stated in more definite terms in Sec. **35.** That section may be read at once, and then studied again in connection with the practical work which precedes it.

28. Increasing and Decreasing Functions.—As a point passes from left to right along the X-axis, x increases algebraically. As a point moves up on the Y-axis, y increases algebraically and as it moves down on the Y-axis, y decreases algebraically. An

increasing function of x is one such that, as x increases algebraically, y, or the function, also increases algebraically. By a **decreasing function of x** is meant one such that, as x increases algebraically, y decreases algebraically. Graphically, an increasing function is indicated by a *rising* curve as a point moves along it from left to right. The power function $y = x^n$ (n positive) is an increasing function of x in the first quadrant and $y = x^{-n}$ ($-n$ negative) is a decreasing function of x in the first quadrant.

The power function $y = x^3$ is an increasing function for all positive and for all negative values of x, while $y = x^2$ is a decreasing function in the second quadrant but an increasing function in the first quadrant. In a case like $y = \pm x^{3/2}$, where y has two values for each positive value of x, it is seen that one of these values increases with x while the other decreases with x.

Exercises

1. Consider the function $y = +x^{1/2}$. As x grows by successive steps of one unit each, does the function grow by increasingly greater and greater steps or not? Is the slope of the curve an increasing or a decreasing function of x?

2. Does the algebraic value of the slope of $xy = 1$ increase with x in the first quadrant?

3. As x changes from -5 to $+5$ does the slope of $y = x^2$ always increase algebraically?

4. Express in the language of mathematics the fact that the curves $y = x^n$, when n is a rational number greater than unity, are *concave upward.*

ANSWER: When n is greater than unity, the slope of the curve increases as x increases.

Express in a similar way the fact that the curves $y = x^{1/n}$ are *concave downward.*

29. The Graph of the Power Function when x^n Has a Coefficient.—If numerical tables be prepared for the equations

$$y = x^2$$

and

$$y = 2x^2$$

then for like values of x each ordinate of the second curve will be two times the corresponding ordinate of the first curve. These

curves are shown in Fig. 36*a*. For each position of P on the curve $y = 2x^2$, $DP = 2DQ$.

It is obvious that the curve

$$y = x^n \tag{1}$$

and the curve

$$y = ax^n \tag{2}$$

are similarly related; the ordinate of any point of the second graph can be made from the corresponding ordinate (*i.e.*, the ordinate

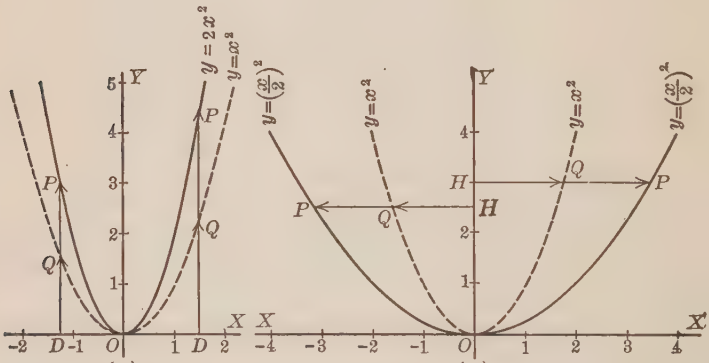


FIG. 36.—(*a*) The curves $y = x^2$ and $y = 2x^2$. (*b*) The curves $y = x^2$ and $y = \left(\frac{x}{2}\right)^2$.

having the same abscissa) of the first graph by multiplying the former by a. If a be positive and greater than unity, this corresponds to *stretching* or elongating all ordinates of Eq. (1) in the ratio $1:a$; if a be positive and less than unity, it corresponds to *contracting* or shortening all ordinates of Eq. (1) in the ratio $1:a$.

For example, the graph of $y = ax^n$ can be made from the graph of $y = x^n$ if the latter be first drawn upon sheet rubber, and if then the sheet be uniformly stretched in the y direction in the ratio $1:a$. If the curve be drawn upon sheet rubber which is already under tension in the y direction and if the rubber be allowed to contract in the y direction, the resulting curve has the equation $y = ax^n$ *where a is a proper fraction or a positive number less than unity.*

The above results are best kept in mind when expressed in a slightly different form. The second of the above two equations, namely, $y = ax^n$, can, of course, be written in the form $y/a = x^n$. Comparing this with the first equation $y = x^n$, it is seen that the second equation is obtained from the first by substituting y/a for y, while, as was seen above, the second graph is obtained from the first by multiplying all the ordinates by a.

In the same manner, changing the equation $y = x^n$ to $y = (x/a)^n$, that is, substituting x/a for x in the equation, multiplies all the abscissas of the curve by a (see Fig. 36*b*).

There is no reason why the letter x could not be used to correspond to vertical distances, and y to horizontal. The essential thing, in a scheme of rectangular coordinates, is simply that the x and y directions be perpendicular. Thus the y direction has no properties or characteristics which the x direction does not also possess. There being no logical distinction between the two coordinates, when one proves that the substitution of y/a for y in the equation multiplies the ordinates of the graph by a, it is also proved, at the same time, that the substitution of x/a for x multiplies the abscissas by a.

The above reasoning is true for the equation of any locus, and we may state the results more generally as follows:

Theorems on Loci

IV. *Substituting x/a for x in the equation of any locus multiplies all of the abscissas of the curve by a.*

V. *Substituting y/a for y in the equation of any locus multiplies all of the ordinates of the curve by a.*

A rough, but essentially correct, way of stating Theorem V, for example, is to say: "dividing y by a in the equation multiplies y by a in the graph." There is a paradoxical element in this statement which should challenge the student's interest, and which should serve as a distinct aid in remembering the theorem. Any tendency which this form of statement may have to make the theorem mysterious is removed by the more extended statement: "When y is multiplied by a on the graph, y must be divided by a in the equation, in order that the equation remain satisfied for the same x."

Having proved these theorems, we are in a position to appreciate the statement made in the first section of this chapter, concerning the relation between $y = x^n$ and $y = ax^n$. The second locus is the first, stretched in the y direction, in the ratio $a:1$.

Exercises

In the following exercises the student should distinguish carefully between such equations as $y = 2x^2$ and $y = (2x)^2$. The former is obtained from $y = x^2$ by substituting $y/2$ for y; the latter by substituting $2x$ for x.

1. Without actual construction, compare the graphs $y = x^2$ and $y = 5x^2$; $y = x^2$ and $y = x^2/2$; $y = 1/x$ and $y = 2/x$; $y = x^3$ and $y = 2x^3$; $y = x^{3/2}$ and $y = x^{3/2}/2$.

2. Without actual construction, compare the graphs $y = x^2$ and $y = (x/2)^2$; $y = x^3$ and $y/2 = x^3$; $y = x^3$ and $y = (x/3)^3$; $y = x^2$ and $y/3 = x^2$.

3. Compare $y^2 = x^3$ and $y^2 = (x/2)^3$; $y^2 = x^3$ and $(y/3)^2 = x^3$; $y^2 = x^3$ and $(y/3)^2 = (x/2)^3$; $y = 1/x^2$ and $y/2 = 1/x^2$.

4. Compare the curves $y = 2x^2$ and $y = 2(x/3)^2$; $3y^2 = x^3$ and $3y^2 = (x/2)^3$; $y^2 = x^3$ and $(2y)^2 = (3x)^3$.

5. Compare $y = x + 3$ and $y = 2(x + 3)$; $y = 2x - 1$ and $y/2 = 2x - 1$; $y = 2x - 1$ and $2y = 2x - 1$.

6. How may the equation $y = af(x)$ be obtained from the equation $y = f(x)$? Without using the above theorems, compare directly the graphs of the two equations, and thus obtain a proof of the general theorem.

30. Change of Unit.—To produce the graph of $y = 10x^2$ from that of $y = x^2$, the stretching of the ordinates in the ratio 1:10 need not actually be performed. If the unit of the vertical scale of $y = x^2$ be taken one-tenth of that of the horizontal scale, and the proper numerical values be placed upon the divisions of the scales, then, obviously, the graph of $y = x^2$ may be used for the graph of $y = 10x^2$. Suitable change in the unit of measure on one or both of the scales of $y = x^n$ is often a very desirable method of representing the more general curve $y = ax^n$.

An interesting example is given in Fig. 37. The period of vibration of a simple pendulum is given by the formula $T = \pi\sqrt{l/g}$. When $g = 981$ cm. per second per second (abbreviated cm./sec.^{2}) this gives $T = 0.1003\sqrt{l}$, which for many purposes is

sufficiently accurate when written $T = 0.10\sqrt{l}$. In this equation T must be in seconds and l in centimeters. Thus when $l = 100$ cm., $T = 1$ sec., so that the graph may be made by drawing the parabola $y = \sqrt{x}$ from the pattern previously made and then attaching the proper numbers to the scales, as shown in Fig. 37.

31. Variation.—The relation between y and x expressed by the equation $y = ax^n$, where n is any positive number, is often expressed by the statement "y varies as the nth power of x," or by the statement "*y is proportional to x^n*." Likewise, the relation $y = a/x^n$, where n is positive, is expressed by the statement "y varies inversely as the nth power of x." The statement "the elongation of a coil spring is proportional to the weight of the suspended mass" tells us

$$y = mx, \qquad (1)$$

where y is the elongation (or increase in length from the natural or unloaded length) of the spring, and x is the weight suspended by the spring. Note especially that the above statement does not give us the value of m. The value of m may readily be determined if the elongation corresponding to a given weight be given. Thus, if a weight of 10 lb. when suspended from the spring produces an elongation of 2 in. in the length of the coil, then, substituting $x = 10$ and $y = 2$ in. Eq. (1),

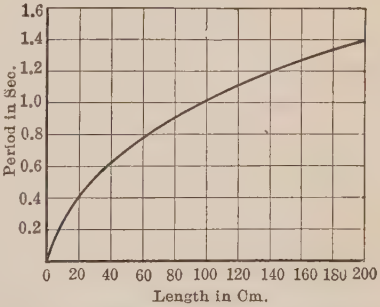


FIG. 37.—Relation of length of a simple pendulum to period of vibration.

$$2 = m10$$

and hence

$$m = \tfrac{1}{5}.$$

If this spring be used in the construction of a spring balance, the length of a division of the uniform scale corresponding to 1 lb. will be $\frac{1}{5}$ in.

A special symbol, $\propto$, is often used to express variation. Thus

$$y \propto \frac{1}{d^2}$$

states that y varies inversely as d^2. It is equally well expressed by

$$y = \frac{k}{d^2},$$

where k is a constant called the **proportionality factor.**

The statements "y varies jointly as u and v," and "y varies directly as u and inversely as v," mean, respectively,

$$y = kuv$$
$$y = \frac{ku}{v}.$$

Thus the area of a rectangle varies jointly as its length and breadth, or

$$A = kLB.$$

If the length and breadth are measured in feet and A in square feet, k is unity. But, if L and B are measured in feet and A in acres, then $k = 1/43{,}560$. If L and B are measured in rods and A in acres, then $k = 1/160$.

From Ohm's law, we say that the electric current in a circuit varies directly as the electromotive force and inversely as the resistance, or

$$C \propto \frac{E}{R}, \text{ or } C = k\frac{E}{R}.$$

The constant multiplier is unity if C be measured in amperes, E in volts, and R in ohms, so that for these units

$$C = \frac{E}{R}.$$

Exercises

1. The original length of a spring is 10 in. The force F necessary to stretch the spring is directly proportional to its elongation, s. (*a*) Find the proportionality factor if a force of 200 lb. will hold the spring at a length of 12 in. (*b*) What force will be required to hold the

spring at a length of 13 in.? (*c*) What force will be required to elongate the spring 1 in.? Note that the elongation is the extension of the length beyond the original length and not the total length after elongation.

SUGGESTION: Since the force F is directly proportional to the elongations, we may write

$$F = ks,$$

where k is the proportionality factor. We have given that F is 200 lb. when s is 2 in.

2. Hooke's law states that the elongation of a steel bar is proportional to the force applied. A bar 500 in. long is stretched to a length of 500.5 in. when a force of 1000 lb. is applied. Find the proportionality factor.

3. Boyle's law states that the volume of a perfect gas at constant temperature varies inversely as the pressure. If volume be measured as cubic feet and pressure as pounds per square inch, find the proportionality factor if the volume be 13 cu. ft. when the pressure is 60 lb. per square inch. What will be the volume of the same gas, according to Boyle's law, if the pressure becomes only 15 lb. per square inch?

32. Illustrations from Science.—Some of the most important laws of natural science are expressed by means of the power function or graphically by means of loci of the parabolic or hyperbolic type.

The linear equation $y = mx$ is, of course, the simplest case of the power function, and its graph, the straight line, may be regarded as the simplest of the curves of the parabolic type. The following illustrations will make clear the importance of the power function in expressing numerous laws of natural phenomena. Later the student will learn of two additional types of fundamental laws of science expressible by two functions entirely different from the power function now being discussed.

The instructor will ask oral questions concerning each of the following illustrations. The student should have in mind the general form of the graph in each case, but should remember that the *law of variation*, or the law of change of value which the functional relation expresses, is the matter of fundamental importance. The graph is useful primarily because it aids in forming a mental picture of the law of variation of the function. The practical

graphing of the concrete illustrations given below will not be done at present, but will be taken up later in Sec. **34.**

1. The pressure of a fluid in a vessel may be expressed in either pounds per square inch or in terms of the height of a column of mercury possessing the same static pressure. Thus we may write

$$p = 0.492h, \tag{1}$$

in which p is pressure in pounds per square inch and h is the height of the column of mercury in inches. The graph is the straight line through the origin of slope 492/1000. The constant 0.492 can be computed from the data that the weight of mercury is 13.6 times that of an equal volume of water and that 1 cu. ft. of water weighs 62.5 lb.

In this and the following equations, it must be remembered that *each letter represents a number*, and that no equation can be used until all the magnitudes involved are expressed in terms of the particular units which are specified in connection with that equation.

2. The velocity of a falling body, which has fallen from a state of rest during the time t, is given by

$$v = 32.2t, \tag{2}$$

in which t is the time in seconds and v is the velocity in feet per second. If t is measured in seconds and v is in centimeters per second, the equation becomes[1] $v = 981t$. In either case the graph is a straight line, but the lines have different slopes.

[1] A full discussion of the process of changing formulas like the ones in the present section into a new set of units should be sought in textbooks on physics and mechanics. The following method is sufficient for elementary purposes. First, write (for the present example) the formula $v = 32.2t$ where v is in ft./sec. and t is in seconds. For *any units of measure that may be used*, there holds a general relation $v = ct$, where c is a constant. To determine what we may call the *dimensions* of c, substitute for all letters in the formula the names of the units in which they are expressed, treating the names as though they were algebraic numbers. From $v = ct$ write, ft./sec. $= c$ sec. Hence (solving for *dimensions* of c), c has dimensions ft./sec.^{2}. Therefore in the given case, we know $c = 32.2$ ft./sec.^{2}. To change to any other units simply substitute equals for equals. Thus 1 ft. $= 30.5$ cm., hence $c = 32.2 \times 30.5$ cm./sec.^{2} $= 981$ cm./sec.^{2}.

To change velocity from mile/hr. to ft./sec. in Eq. (19) below, we have $R = 0.003V^2$ where R is in lb./sq.-ft. and V is in mile/hr. Write the general formula $R = cV^2$. The dimensions of c are (lb./ft.^{2}) $\div$ (mile^2/hr.^{2}) or (lb./ft.^{2}) $\times$ (hr.^{2}/mile^2). In the given case we have the *value* of $c = 0.003$ (lb./ft.^{2}) $\times$ (hr.^{2}/mile^2). To change V to ft./sec., substitute equals for equals, namely 1 hr. $= 3600$ sec., 1 mile $= 5280$ ft., which gives the formula $R = 0.00139V^2$, where V is expressed as ft./sec. and R is expressed as lb./ft.^{2}. Note that 1 mile/hr. $= \frac{3}{2}$ ft./sec. *approximately.*

3. The space traversed by a falling body is given by

$$s = \tfrac{1}{2}gt^2, \tag{3}$$

or in English units (s in feet and t in seconds)

$$s = 16.1t^2. \tag{4}$$

4. The velocity of the body falling from the height h is

$$v = \sqrt{2gh} = \sqrt{64.4h}. \tag{5}$$

The resistance of the air is not taken into account in Eqs. (2) to (5). The formula equivalent to Eq. (5),

$$\tfrac{1}{2}mv^2 = mgh, \tag{6}$$

where m is the mass of the body, expresses the equivalence of $\frac{1}{2}mv^2$, the kinetic energy of the body, and mgh, the work done by the force of gravity mg, working through the distance h.

5. The intensity of the attraction exerted on a unit mass by the sun or by any planet varies inversely as the square of the distance from the center of mass of the attracting body. If r stand for that distance and if f be the force exerted on unit mass of the attracted body, then

$$f = \frac{m}{r^2}. \tag{7}$$

The constant m is the value of the force when r is unity.

6. The formula for the horsepower transmissible by cold-rolled shafting is

$$H = \frac{d^3N}{50}, \tag{8}$$

where H is the horsepower transmitted, d the diameter of the shaft in inches, and N the number of revolutions per minute.

The rapid increase of this function (as the cube of the diameter) accounts for some interesting facts. Thus, doubling the size of the shaft operating at a given speed increases eightfold the amount of power that can be transmitted, while the weight of the shaft is increased but fourfold.

If H be constant, N varies inversely as d^3. Thus, an old-fashioned 50-hp. overshot water wheel making 3 revolutions per minute requires about a 9-in. shaft, while a DeLaval 50-hp. steam turbine making 16,000 revolutions per minute requires a turbine shaft but little over $\frac{1}{2}$ in. in diameter.

7. The period of the simple pendulum is

$$T = \pi\sqrt{\frac{l}{g}}, \tag{9}$$

where T is the time of one swing in seconds, l the length of the pendulum in feet, and $g = 32.2$ ft./sec.^{2}, approximately.

8. The centripetal force on a particle of weight W lb., rotating in a circle of radius R ft., at the rate of N revolutions per second is

$$F = \frac{4\pi^2 WRN^2}{g}, \tag{10}$$

or, if $g = 32.16$ ft./sec.^{2},

$$F = 1.2276WRN^2, \tag{11}$$

where F is measured in pounds. If N be the number of revolutions per minute, then

$$F = \frac{4\pi^2 WRN^2}{3600g} \tag{12}$$

$$= 0.000341WRN^2. \tag{13}$$

9. An approximation formula for the indicated horsepower required for a steamboat is

$$I.hp. = \frac{S^3 D^{2/3}}{C}, \tag{14}$$

where S is speed in knots, D is displacement in tons, and C is a constant appropriate to the size and model of the ship to which it is applied. The constant ranges in value from about 240, for finely shaped boats to 200 for fairly shaped boats.

10. Boyle's law for the expansion of a gas maintained at constant temperature is

$$pv = C, \tag{15}$$

where p is the pressure and v the volume of the gas, and C is a constant. Since the density of a gas is inversely proportional to its volume, the above equation may be written in the form

$$p = c\rho, \tag{16}$$

in which ρ is the density of the gas.

11. The flow of water over a trapezoidal weir is given by

$$q = 3.33Lh^{3/2}, \tag{17}$$

where q is the quantity in cubic feet per second, L is the length of the weir[1] in feet, and h is the head of water on the weir, in feet.

[1] The instructor is expected to explain the meaning of the terms here used.

12. The physical law holding for the adiabatic expansion of air, that is, the law of expansion holding when the change of volume is not accompanied by a gain or loss of heat,[1] is expressed by

$$p = c\rho^{1.408}. \tag{18}$$

This is a good illustration of a power function with fractional exponent. The graph is not greatly different from the semi-cubical parabola

$$y = cx^{3/2}.$$

13. The pressure or resistance of the air upon a flat surface perpendicular to the current is given by the formula

$$R = 0.003V^2, \tag{19}$$

in which V is the velocity of the air in miles per hour and R is the resulting pressure upon the surface in pounds per square foot. According to this law, a 20-mile wind would cause a pressure of about 1.2 lb. per square foot upon the flat surface of a building. One foot per second is equivalent to about $\frac{2}{3}$ mile per hour, so that the formula when the velocity is given in feet per second becomes:

$$R = 0.0013V^2. \tag{20}$$

14. The power used to drive an aeroplane may be divided into two portions. One portion is utilized in overcoming the resistance of the air to the onward motion. The other part is used to sustain the aeroplane against the force of gravity. The first portion does "useless" work—work that should be made as small as possible by the shapes and the sizes of the various parts of the machine. The second part of the power is used to form continuously anew the wave of compressed air upon which the aeroplane rides. Calling the total power[2] P, the power required to overcome the resistance P_r, and that used to sustain the aeroplane P_s, we have

$$P = P_r + P_s. \tag{21}$$

We learn from the theory of the aeroplane that P_r varies as the cube of the velocity, while P_s varies inversely as V, so that

$$P_r = cV^3 \tag{22}$$

and

$$P_s = \frac{k}{V}. \tag{23}$$

[1] Note that when a vessel containing a gas is insulated by a non-conductor of heat, so that no heat can enter or escape from the vessel, the *temperature* of the gas will rise when the gas is compressed and fall when it is expanded. Adiabatic expansion may be thought of, therefore, as taking place in an *insulated* vessel.

[2] Power (= work done per unit time) is measured by the unit *horsepower*, which is 550 ft.-lb. per second.

Thus at high velocity less and less power is required to sustain the aeroplane but more and more is required to overcome the frictional resistance of the medium. The law expressed by Eq. (23) that less and less power is required to sustain the aeroplane as the speed is increased is known as Langley's law. From this law Langley was convinced that artificial flight was possible, for the whole matter seemed to depend primarily upon getting up sufficient speed. It is

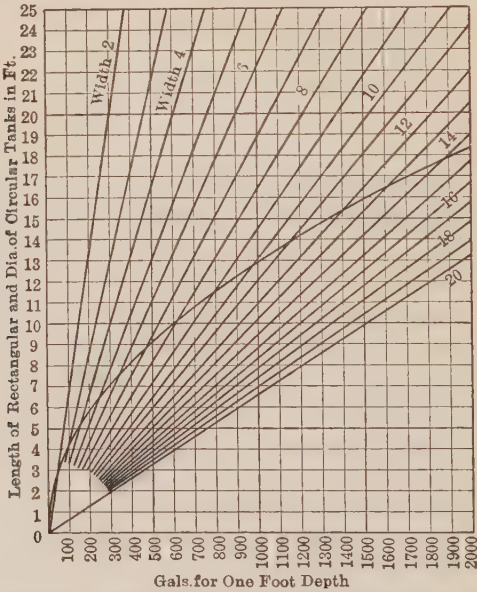


FIG. 38.—Capacity of rectangular and circular tanks per foot of depth.

really this law that makes the aeroplane possible. An analogous case is the well-known fact that the faster a person skates the thinner the ice necessary to sustain the skater. In this case part of the energy of the skater is continually forming anew on the thin ice the wave of depression which sustains the skater, while the other part overcomes the frictional resistance of the skates on the ice and the resistance of the air.

15. The capacity of cast-iron pipe to transmit water is often given by the formula

$$q^{1.86} = 1.68hd^{3.25}, \tag{24}$$

in which q is the quantity of water discharged in cubic feet per second, d is the diameter of the pipe in feet, and h is the loss of head measured in feet of water per 1000 linear ft. of pipe. This is a good illustration of the equation of a parabolic curve with complicated fractional exponents. The curve very roughly approximates the locus of the equation

$$y = c\sqrt{h}\, x^{3\frac{1}{2}}. \tag{25}$$

16. The contents in gallons of a rectangular tank per foot of depth, b ft. wide and l ft. long, is

$$q = 7.5bl. \tag{26}$$

The contents in gallons per foot of depth of a cylindrical tank d ft. in diameter is

$$q = \frac{7.5\pi d^2}{4}. \tag{27}$$

Figure 38 shows the graph of Eq. (26) for various values of b and also shows to the same scale the graph of Eq. (27).

33. Rational and Empirical Equations.—A number of the formulas given above are capable of demonstration by means of theoretical considerations only. Such, for example, are Eqs. (1), (2), (3), (4), (5), (7), (8), (9), (10), etc., although the constant coefficients in many of these cases were experimentally determined. Formulas of this kind are known in mathematics as **rational equations.** On the other hand, certain of the above formulas, especially Eqs. (14), (17), (19), (22), (23), (24), including not only the constant coefficients but also the *law of variation of the function itself*, are known to be true only as the result of experiment. Such equations are called **empirical equations.** Such formulas arise in the attempt to express by an equation the results of a series of laboratory measurements.

For example, the density of water (that is, the mass per cubic centimeter or the weight per cubic foot) varies with the temperature of the water. A large number of experimenters have prepared accurate tables of the density of water for wide ranges of temperature Centigrade, and a number of very accurate empirical formulas have been ingeniously devised to express the results, of which the following four equations are samples:

Empirical formulas *for the density, d, of water in terms of temperature Centigrade, θ.*

$$d = 1 - \frac{96(\theta - 4)^2}{10^7}. \tag{1}$$

$$d = 1 - \frac{93(\theta - 4)^{1.982}}{10^7}. \tag{2}$$

$$d = 1 - \frac{6\theta^2 - 36\theta + 47}{10^6}. \tag{3}$$

$$d = 1 + \frac{0.485\theta^3 - 81.3\theta^2 + 602\theta - 1118}{10^7}. \tag{4}$$

Exercises

1. Among the power functions named in the above illustrations, pick out examples of increasing functions and of decreasing functions.

2. Under the same difference of head or pressure, show by Eq. (24) that an 8-in. pipe will transmit much more than double the quantity of water per second that can be transmitted by a 4-in. pipe.

3. Wind velocities during exceptionally heavy hurricanes on the Atlantic coast are sometimes over 140 miles per hour. Show that the wind pressure on a flat surface during such a storm is about fifty times the amount experienced during a 20-mile wind.

4. Show that for wind velocities of 10, 20, 40, 80, 160 miles per hour (varying in geometrical progression with ratio 2), the pressure exerted on a flat surface is 0.3, 1.2, 4.8, 19.2, 76.8 lb. per square foot respectively (varying in geometrical progression with ratio 4).

5. A 300-hp. DeLaval turbine makes 10,000 r.p.m. Find the necessary diameter of the propeller shaft.

6. A railroad switch target bent over by the wind during a tornado in Minnesota indicated an air pressure due to a wind of 600 miles per hour. Show that the equivalent pressure on a flat surface would be 7.5 lb. per square inch.

7. Show that a parachute 50 ft. in diameter and weighing 50 lb. will sustain a man weighing 205 lb. when falling at the rate of 10 ft. per second.

SUGGESTION: Use approximate value $\pi = 22/7$ in finding area of parachute from the formula for the circle, πr^2, and use Eq. (20) above.

8. Show that empirical Eqs. (1) and (2) for the density of water reduce to a power function if the origin be taken at $\theta = 4$, $d = 1$.

34. Practical Graphs of Power Functions.—The graphs of the power function

$$y = x^2,\ y = x^3,\ y = \frac{1}{x},\ y = x^{3/2},\ \text{etc.} \qquad (1)$$

can, of course, be made the basis of the laws concretely expressed by Eqs. (1) to (27) of Sec. **32.** If, however, the graph of a scientific formula is to serve as a numerical table of the function for actual use in practical work, then there is much more labor in the proper construction of the graph than the mere plotting of the abstract mathematical function. The size of the unit to be selected, the range over which the graph should extend, the permissible course of the curve, become matters of practical importance.

If the apparent slope[1] of a graph departs too widely from $+1$ or -1, it is desirable to make an abrupt change of unit in the vertical or the horizontal scale, so as to bring the curve back to a desirable course, for it is obvious that numerical readings can best be taken from a curve when it crosses the rulings of the coordinate paper at apparent slopes differing but little from ± 1.

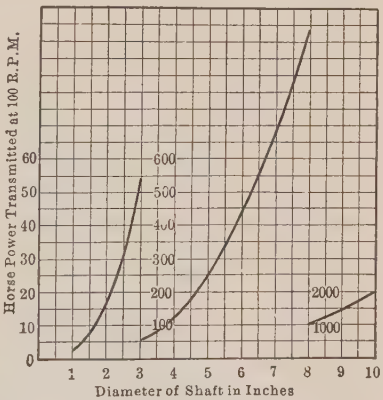


FIG. 39.—Capacity at 100 R.P.M. of cold-rolled shafting to transmit power.

The above suggestions in practical graphing are illustrated by the following example:

Graph the formula (Eq. (8), Sec. **32**), for the horsepower transmissible by cold-rolled shafting,

$$H = \frac{d^3N}{50}, \qquad (2)$$

in which d is the diameter in inches and N is the number of revolutions per minute. The formula is of interest only for the range of

[1] Of course the *real* slope of a curve is independent of the scales used. By *apparent* slope = 1 is meant that the graph cuts the ruling of the squared paper at about 45°.

d between 0 and 24 in., as the dimensions of ordinary shafting lie well within these limits. Likewise, one would not ordinarily be interested in values of N except those lying between 10 and 3000 r.p.m. Figure 39 shows a suitable graph of this formula for the range $1 < d < 10$ for the fixed value of $N = 100$. In order properly to graph this function, three different scales have been used for the ordinate H, so that the slope of the curve may not depart too widely from unity.

If similar graphs be drawn for $N = 200$, $N = 300$, $N = 400$, etc., a set of parabolic curves is obtained from which the horsepower of shafting for various speeds of rotation as well as for various diameters may be obtained at once. A set of curves systematically constructed in a manner similar to that just described is often called a **family of curves.** Figure 38 shows a family of straight lines expressing the capacity of rectangular tanks corresponding to the various widths of the tanks.

Inasmuch as many of the formulas of science are used only for positive values of the variables, it is only necessary in these cases to graph the function in the first quadrant. For such problems the origin may be taken at the lower left corner of the coordinate paper so that the entire sheet becomes available for the curve in the first quadrant.

The illustrations of Sec. **32** are sufficient to make clear the importance in science of the functions now being discussed. The following exercises give further practice in the useful application of the properties of the functions.

Exercises

The graphs for the following problems are to be constructed upon rectangular coordinate paper. The instructions are for centimeter paper (form $M1$) ruled into 20- by 25-cm. squares. On other paper use ½ in. in place of 1 cm. In each case the units for abscissa and for ordinates are to be so selected as best to exhibit the functions, considering both the workable range of values of the variables and the suitable slope of the curves.

The student should read Sec. **14** a second time before proceeding with the following exercises, giving especial care to instructions 4, 5, and 6 of that section.

1. Classify the graphs of Eqs. (1) to (27), Sec. **32** as to parabolic or hyperbolic type.

2. Graph the formula $v^2 = 2gh$, or $v = \sqrt{2gh} = 8.02h^{\frac{1}{2}}$ if h range between 1 and 100, the second and the foot being the units of measure (see Eq. (5), Sec. **32**).

The following table of values is readily obtained:

h	1	5	10	20	30	40	50	60	70	80	90	100
v	8.02	17.9	25.3	35.8	43.9	50.7	56.7	62.1	67.1	71.7	76.0	80.2

Use 2 cm. = 10 ft. as the horizontal unit for h, and 2 cm. = 10 ft. per second as the vertical unit for v. The graph is then readily constructed without change of unit or other special expedient.

3. Graph the formula $q = 3.33Lh^{\frac{3}{2}}$ for $L = 1$, and for $h = 0$, 0.1, 0.2, 0.3, 0.4, 0.5 (see Eq. (17), Sec. **32**). Use 4 cm. = 0.1 for horizontal unit for h and 2 cm. = 0.1 for vertical unit for q.

4. Draw a curve showing the indicated horsepower of a ship $i.hp. = S^3D^{\frac{2}{3}}/C$ for $C = 200$ if the displacement $D = 8000$ tons, and for the range of speeds $S = 10$ to $S = 20$ knots (see Eq. (14), Sec. **32**).

For the vertical unit use 1 cm. = 1000 hp. and for the horizontal unit use 2 cm. = 1 knot. Call the lower left-hand corner of the paper the point ($S = 10$, $i.hp. = 0$).

5. From the formula expressing the centripetal force in pounds of a rotating body,

$$F = 0.000341WRN^2,$$

draw a curve showing the total centripetal force sustained by a 36-in. automobile tire weighing 25 lb., for all speeds from 10 to 40 miles per hour (see Eq. (13), Sec. **32**).

Miles per hour must first be converted into r.p.m. by dividing 5280 by the circumference of the tire and then dividing the result by 60. This gives

$$1 \text{ mile per hour} = 9\tfrac{1}{3} \text{ r.p.m.}$$

If V be the speed in miles per hour the formula for F becomes

$$F = 0.000341(1.5)25(9\tfrac{1}{3})^2V^2 = 1.11V^2.$$

For horizontal scale let 4 cm. = 10 miles per hour and for the vertical scale let 1 cm. = 100 lb.

6. Draw a curve from the formula $f = m/r^2$, showing the acceleration of gravity due to the earth at all points between the surface of the earth and a point 240,000 miles (the distance to the moon) from the center, if $f = 32.2$ when the radius of the earth = 4000 miles.

It is convenient in constructing this graph to take the radius of the earth as unity, so that the graph will then be required of $f =$

$32.2/r^2$ from $r = 1$ to $r = 60$. In order to construct a suitable curve several changes of units are desirable (see Fig. 40). One centimeter represents one radius (4000 miles) from $r = 0$ to $r = 10$, after which the scale is reduced so that 1 mm. represents one radius. In the vertical direction the scale is 4 cm. = 10 ft. per second^2 for $0 < r < 5$, 4 cm. = 1 ft. per second^2 for $5 < r < 10$, and 4 cm. = 0.1 ft. per second^2 for $10 < r < 60$. Even with these four changes of units just used the first and third curves are somewhat steep. The student can readily improve on the scheme of Fig. 40 by a better selection of units.

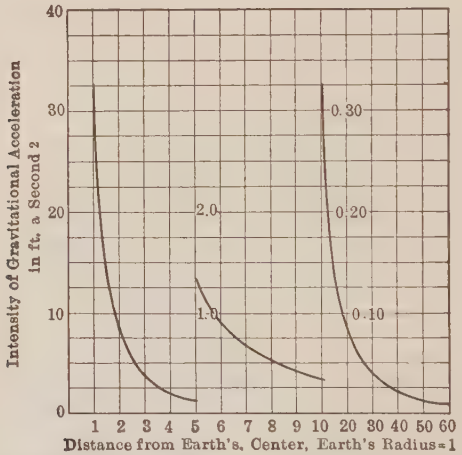


FIG. 40.—Gravitational acceleration at various distances from the earth's center. The moon is distant approximately 60 earth's radii from the center of the earth.

35. The Law of the Power Functions.—Sufficient illustrations have been given to show the fundamental character of the power function as an expression of numerous laws of natural phenomena. How may a functional dependence of this sort be expressed in words? If a series of measurements are made in the laboratory, so as to produce a numerical table of data covering certain phenomena, how can it be determined whether or not a power function can be written down which will express the law (that is, the function) defined by the numerical table of laboratory results?

The answers to these questions are readily given. Consider first the law of the falling body

$$s = 16.1t^2. \tag{1}$$

Make a table of values for values of $t = 1, 2, 4, 8, 16$ sec., as follows:

t	1	2	4	8	16
s	16.1	64.4	257.6	1030.4	4121.6

The values of t have been so selected that t increases by a fixed multiple, that is, each value of t in the sequence is twice the preceding value. From the corresponding values of s it is observed that s also increases by a fixed multiple, namely, 4.

Similar conclusions, obviously, hold for any power function. Take the general case

$$y = ax^n, \tag{2}$$

where n is any exponent, positive, negative, integral, or fractional. Let x change from any value x_1 to a multiple value mx_1 and call the corresponding values of y, y_1 and y_2. Then we have

$$y_1 = ax_1^n, \tag{3}$$

and

$$y_2 = a(mx_1)^n = am^n x_1^n. \tag{4}$$

Divide the members of Eq. (4) by the members of Eq. (3) and we have

$$\frac{y_2}{y_1} = m^n. \tag{5}$$

That is, if x in any power function change by the fixed multiple m, then the value of y will change by a fixed multiple m^n. Thus the law of the power function may be stated in words in either of the two following forms:

In any power function of x, if x change by a fixed multiple, y will change by a fixed multiple also.

In any power function of x, if x increase by a fixed per cent, the function will increase or decrease by a fixed per cent also.

This test may readily be applied to laboratory data to determine whether or not a power function can be set up to represent as a formula the data in hand. To apply this test, select, at several places in one column of the laboratory data, pairs of numbers

which change by a selected fixed per cent, say 10, 20, or any convenient per cent. Then the corresponding pairs of numbers in the other column of the table must also be related by a fixed per cent (of course, not, in general, the same as the first-named per cent), provided the functional relation is expressible by means of a power function. If this test does not succeed, then the function in hand is not a power function.

Since the fixed per cent for the function is m^n if the fixed per cent for the variable be m, the possibility of determining n exists, since the table of laboratory data must yield the numerical values of both m and m^n.

36. Translation of Any Locus.—In studying the equation of the straight line it was noted that the substitution, in the equation, of of $(y - b)$ for y translated the locus b units in the positive y direction. This is a special case of a general theorem which will now be proved.

The equation $y = f(x)$ states that there is a functional relation between y and x, but does not tell what that relation is. This equation may, accordingly, be viewed as the equation of *any* curve. Let us compare the curve $y = f(x)$ and the curve $y = f(x) + b$. Using the same value of x in the two equations, the value of y in the second equation is b units larger than the value of y as given by the first equation. Since this is true, whatever the value of x, the second curve is exactly similar in shape to the first, but is located b units higher. The second equation, on the other hand, is obtained from the first by substituting $(y - b)$ for y. We have thus proved the following theorems:

THEOREMS ON LOCI

VI. *If* $(x - a)$ *be substituted for* x *throughout any equation, the locus is translated a distance* a *in the* x *direction.*

VII. *If* $(y - b)$ *be substituted for* y *in any equation, the locus is translated the distance* b *in the* y *direction.*

These statements are perfectly general; if the signs of a and b are negative, so that the substitutions for x and y are of the form $x + a'$ and $y + b'$, respectively, then the translations are to the *left* and *down* instead of to the *right* and *up*.

Sometimes the motion of translation may seem to be disguised by the position of the terms a or b. Thus the locus $y = 3x + 5$ is the same as the locus $y = 3x$ translated upward the distance 5, for the first equation is really $y - 5 = 3x$, from which the conclusion is obvious.

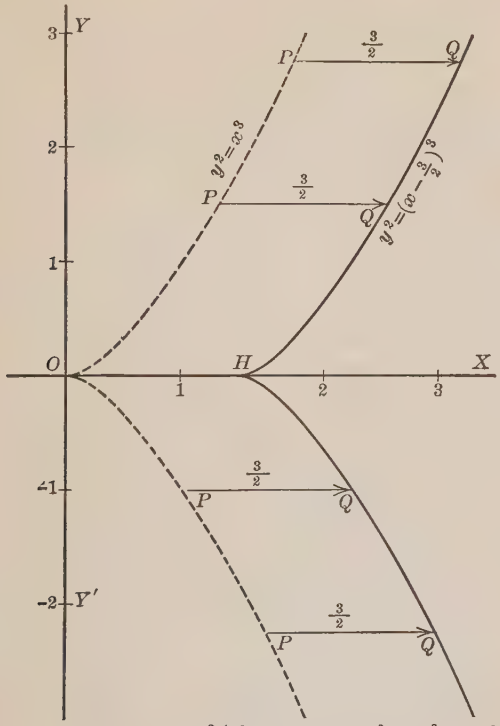


FIG. 41.—The curve $y^2 = (x - 3/2)^3$ is the curve $y^2 = x^3$ translated to the right $3/2$ units.

In Fig. 41 is shown an illustration of the use of one of these theorems.

Exercises

The student is not required to draw the curves in Exercises 1 to 8 below, but is expected to make the comparisons by means of the theorems on loci given above.

1. Compare the curves:

(*a*) $y = 2x$ and $y = 2(x - 1)$.
(*b*) $y = x^3$ and $y = (x - 4)^3$.
(*c*) $y = x^3$ and $y - 3 = x^3$.
(*d*) $y = x^{3/2}$ and $y = (x - 5)^{3/2}$.
(*e*) $y = 5x^2$ and $y = 5(x + 3)^2$.
(*f*) $y = 2x^3$ and $y = 2(x - k)^3$.
(*g*) $y = 2x^3$ and $y = 2x^3 + k$.
(*h*) $y + 7 = x^2$ and $y = x^2$ and $y - 7 = x^2$.
(*i*) $3y^2 = 5x^3$ and $3(y - b)^2 = 5(x - a)^3$.

2. Compare the curves:

(*a*) $y = x^3$ and $y = \left(\frac{x}{2}\right)^3$.
(*b*) $y = x^3$ and $y = \frac{x^3}{8}$.
(*c*) $y = x^3$ and $\frac{y}{2} = x^3$.
(*d*) $y = x^3$ and $y = 2x^3$.
(*e*) $y^2 = 3x^3$ and $\left(\frac{y}{5}\right)^2 = 3\left(\frac{x}{7}\right)^3$.
(*f*) $y^2 = x^3$ and $y^2 = (3x)^3$.
(*g*) $y = x^2$ and $y = 4x^2$ (NOTE: explain in two ways).
(*h*) $y = x^3$ and $2y = x^3$ and $y = 27x^3$.

3. Translate the locus $y = 2x^3$; (*a*) 3 units to the right; (*b*) 4 units down; (*c*) 5 units to the left.

4. Elongate threefold in the x direction the loci: (*a*) $y^2 = x$; (*b*) $3y = x^3$; (*c*) $y^2 = 2x^3$; (*d*) $y = 2x + 7$.

5. The loci named in Exercise 4 have their ordinates shortened in the ratio 2:1; write their equations.

6. Show that $y = \frac{1}{x + b}$ and $y = \frac{1}{x - b}$ are hyperbolas.

7. Show that $y = \frac{x}{x + b}$ is an hyperbola.

NOTE: Divide the numerator by the denominator, obtaining the equation $y = 1 - \frac{b}{x + b}$, or $y - 1 = -\frac{b}{x + b}$.

8. Show that

$$y = \frac{x + a}{x + b},$$

or

$$y = 1 + \frac{a - b}{x + b},$$

is an hyperbola, namely, the curve $xy = a - b$ translated to a new position.

9. Sketch: (*a*) $y = \dfrac{x}{x-1}$; (*b*) $y = \dfrac{x+3}{x+1}$; (*c*) $y = \dfrac{3x+5}{x+1}$.

Each of these curves is formed by stretching and translating the rectangular hyperbola $y = 1/x$. The curves may be sketched by determining, in each case, the amount of the stretch and of the translation. Determine, from the equations as given, the X- and Y-intercepts of each curve, and check the sketches by means of these values.

10. In Exercise 9(*a*), the denominator vanishes for what value of x? How is this shown on the sketch? In Exercise 9(*b*), what value does y approach as x becomes very large? How is this shown on the sketch?

11. Referring to Exercise 9(*c*), discuss the special values: $x = -\frac{5}{3}$, $x = -1$, $y = 5$, $y = 3$.

12. Show how the graph for $y = x^2 + 4x + 5$ may be obtained from the graph for $y = x^2$.

Hint: $y = x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x + 2)^2 + 1$, or $y - 1 = (x + 2)^2$. Thus, the graph for $y = x^2 + 4x + 5$ may be obtained by translating the graph for $y = x^2$, 1 unit up and 2 units to the left.

13. Sketch the curves for:

(*a*) $y = x^2 + 4x + 4$.
(*b*) $y = x^2 + 6x + 10$.
(*c*) $y = x^2 + 2x - 3$.
(*d*) $y = 4x^2 + 4x + 1$.
(*e*) $y = 4x^2 + 2x - 1$.
(*f*) $y = 4x - x^2$.
(*g*) $y = 6x - x^2$.
(*h*) $y = x^2 + 3x - 1$.
(*i*) $y = 2x^2 + 3x$.
(*j*) $y = 3x - 2x^2$.
(*k*) $y^2 = x + 1$.

14. Which of the curves of Exercise 13 pass through the origin?

15. Sketch:

(*a*) $x^2 + y^2 = 1$.
(*b*) $x^2 + y^2 = 4$.
(*c*) $x^2 + (y - 1)^2 = 1$.
(*d*) $(x - 1)^2 + y^2 = 4$.
(*e*) $(x + 1)^2 + (y - 2)^2 = 5$.
(*f*) $x^2 + 2x + y^2 = 3$.

37. Shearing Motion.—Let the dotted curve $P_1'OP_1$ (Fig. 42) be the graph of the semi-cubical parabola $y = x^{3/2}$ and OP_2 the graph of $y = x$. The graph $P'OP$ is constructed by taking its ordinate, for any value of the abscissa, equal to the (algebraic) sum of the ordinates of the two given curves. Thus, $DP = DP_1 + DP_2$ and $DP' = DP_1' + DP_2$. The equation of $P'OP$ is $y = x^{3/2} + x$, since $DP_1 = x^{3/2}$ and $DP_2 = x$.

Exercises

1. From the curves for $y = x^2$ and $y = \frac{1}{2}x$, sketch $y = x^2 + \frac{1}{2}x$.

2. From the curves for $y = x^2$ and $y = -\frac{1}{2}x$, sketch $y = x^2 - \frac{1}{2}x$.

3. From the curves for $y = -x^2$ and $y = x$, sketch $y = x - x^2$.

4. From the curves for $y = 1/x$ and $y = x$, sketch $y = 1/x + x$.

5. From the curves for $y = 1/x$ and $y = x$, sketch $y = x - 1/x$.

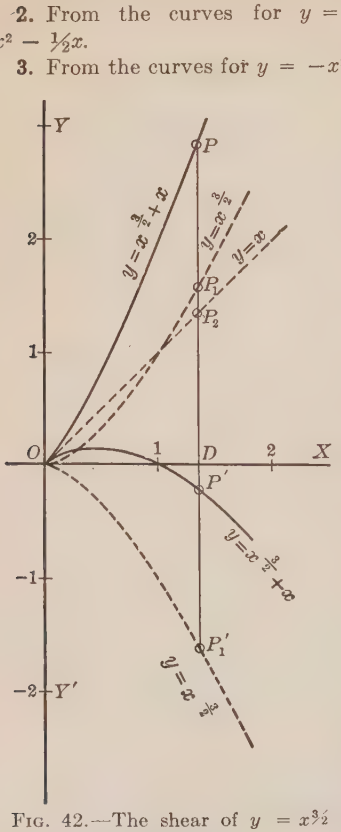


FIG. 42.—The shear of $y = x^{3/2}$ in the line $y = x$.

38. General Case.—Consider the production of the curve

$$y = f(x) + mx \qquad (1)$$

from the curve

$$y' = f(x) \qquad (2)$$

and the straight line

$$y'' = mx. \qquad (3)$$

Graphically, the curve (1) is seen to be formed by the addition of the ordinates of the straight line $y'' = mx$ to the corresponding ordinates of $y' = f(x)$. Thus, in Fig. 43, the graph of the function $x^3 + x$ is made by adding the corresponding ordinates of $y = x^3$ and $y = x$. Mechanically, this might be done by drawing the curve on the edge of a pack of cards (see Fig. 44), and then slipping the cards over each other uniform amounts. The change of the shape of a body, or the strain of a body, here illustrated is called **lamellar motion** or **shearing motion.** It is a form of motion of very great importance.

We shall speak of the locus $y = f(x) + mx$ as *the shear of the curve* $y = f(x)$ *in the line* $y = mx$.

Theorems on Loci

VIII. *The addition of the term mx to the right side of $y = f(x)$ shears the locus $y = f(x)$ in the line $y = mx$.*

The locus

$$y = ax^3 + mx + b$$

is made from $y = x^3$ by a combination of, first, a uniform elongation a; second, a shearing motion m; and, third, a translation b.

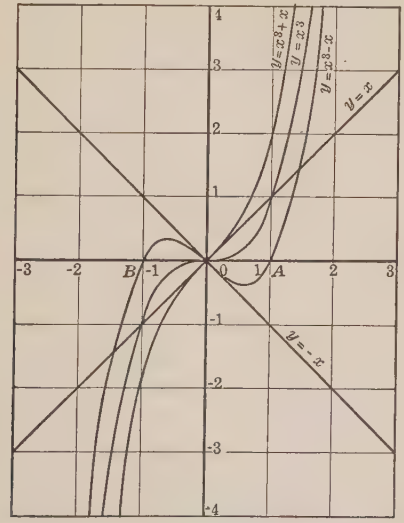


Fig. 43.—The shear of the cubical parabola $y = x^3$ in the line $y = x$, and also in the line $y = -x$.

Either motion may be changed in sense by changing the sign of a, m, or b, respectively.

The student may easily show that the effect of a shearing motion upon the straight line $y = mx + b$ is merely a rotation about the fixed point $(0, b)$. The line is really *stretched* in the direction of its own length, but this does not change the shape of the line, nor does it change the line geometrically. A *line segment* (that is, a line of finite length) would be modified, however.

The parabola $y = x^2$ is transformed under a shearing motion in a most interesting way. For, after shear, $y = x^2$ becomes

$$y = x^2 + 2mx, \tag{4}$$

where, for convenience, the amount of the shearing motion is

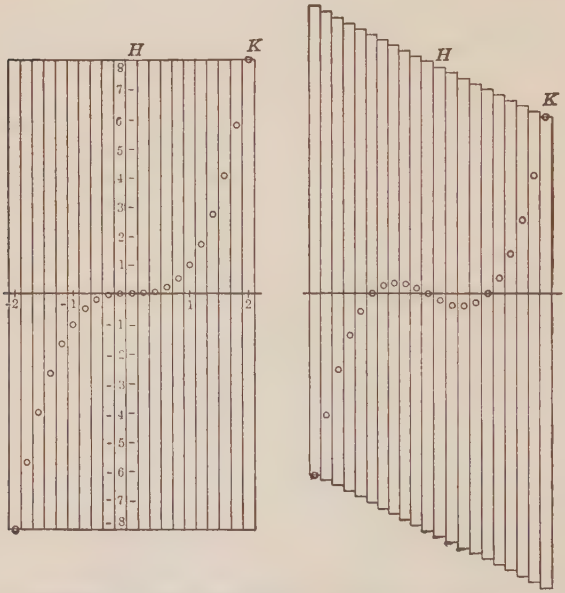


FIG. 44.—Shearing motion illustrated by slipping of the members of pack of cards.

represented by $2m$ instead of by m. Writing this in the form

$$y = x^2 + 2mx + m^2 - m^2,$$

or

$$y = (x + m)^2 - m^2,$$
$$y + m^2 = (x + m)^2, \tag{5}$$

we see that the locus of Eq. (4) can be made from the parabola $y = x^2$ by translating the curve *to the left* the amount m and *down* the amount m^2 (see Fig. 45).

Shearing motion, therefore, rotates the straight line and translates the parabola. The effect on other curves is much more complicated, as is seen from Figs. 42 to 44.

The parabola $y = x^2$ after shear is identical in size and shape with $y = x^2 + mx + b$. Likewise, $y = ax^2 + bx + c$ is a parabola differing only in position from $y = ax^2$.

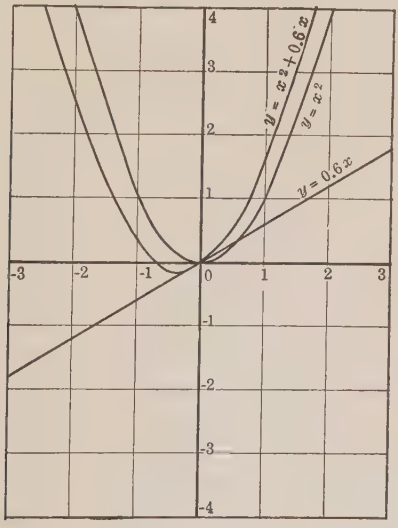


FIG. 45.—The shear of $y = x^2$ in the line $y = 0.6x$.

Note that if the curve $4y^2 = x^3$ is to be sheared in the straight line $2y = 3x$, the equations must first be put in the form $y = \frac{1}{2}x^{\frac{3}{2}}$ and $y = \frac{3}{2}x$, and the resulting equation is $y = \frac{1}{2}x^{\frac{3}{2}} + \frac{3}{2}x$; that is, the *ordinates* are to be added, not some function of the ordinates.

Exercises

1. Explain how the curve $y = x^3 + 2x$ may be made from the curve $y = x^3$. How can the curve $y = 2x^3 + 3x$ be made from the curve $y = 2x^3$?

2. Find the coordinates of the lowest point of $y = x^2 - 4x$, that is, put this equation in the form $y - b = (x - a)^2$.

3. Compare the curves $y = x^3 + 2x$ and $y = x^3 - 2x$. (Do not *draw* the curves.)

4. Explain how the curve $y = 1/x + 2x$ may be formed from the curves $y = 1/x$ and $y = 2x$.

39. Rotation of a Locus.—The only simple type of displacement of a locus not yet considered is the rotation of the locus about the origin O. This will be taken up in the next chapter.

40. Roots of Functions.—The **roots,** or **zeros,** of a function are the values of the argument for which the corresponding value of the function is zero. Thus, 2 and 3 are roots of the function $x^2 - 5x + 6$, for substituting either number for x causes the function to become zero. The roots of $x^2 - x - 6$ are $+3$ and -2. The roots of $x^3 - 6x^2 + 11x - 6$ are 1, 2, 3.

The word root, used in this sense, has, of course, an entirely different significance from the same word in "square root," "cube root," etc. But the roots of the function $x^2 - 5x - 6$ are also the roots of the *equation* $x^2 - 5x - 6 = 0$.

In the graph of the cubic function $y = x^3 - x$ in Fig. 43, the curve crosses the X-axis at $x = -1$, $x = 0$, and $x = 1$. These are the values of x that make the function $x^3 - x$ zero, and are, of course, the roots of the function $x^3 - x$. No matter what the function may be, it is obvious that *the intercepts on the X-axis of the curve* $y = f(x)$, as OA, OB (Fig. 43), *must represent the roots of* $f(x)$.

Exercises

1. From the curve $y = x^2$ sketch the curves $y - 4 = x^2$; $y = 4x^2$; $4y = x^2$; $y = (x - 4)^2$.

2. Sketch $y = x^3/2$; $y = x^3 - \frac{1}{4}$; $y = x^3/2 - 4$; $y = (x - 3)^2/2$.

3. Sketch the curves $y = x^{\frac{1}{2}}$; $y = x^{\frac{1}{3}}$; $y = 2x^{\frac{1}{2}}$; $y = (x - 2)^{\frac{1}{2}}$; $y - 2 = (x - 2)^{\frac{1}{2}}$ and $y = (x - 3)^{\frac{1}{3}}$.

4. Sketch the curves $y^2 = (x - 3)^3$; $(y - 2)^2 = x^3$; and $(y - 2)^2 = (x - 3)^3$.

5. Graph $y_1 = x$ and $y_2 = x^3$ and thence $y = x + x^3$.

6. Find the X-intercepts for the following:

(*a*) $y = x^2 + 2x - 3$.
(*b*) $y = x^3 - x$.
(*c*) $y = 2x^2 + x - 3$.

7. Find the roots of the following functions:

(*a*) $x^2 - 6x + 8$. (*d*) $2x^2 - 5x + 2$.
(*b*) $x^2 + x - 2$. (*e*) $6x^2 + x - 1$.
(*c*) $x^2 - x - 6$. (*f*) $x^3 + x^2 - 2x$.

41. As was shown in Sec. **20,** the coordinates of the point or points of intersection of two loci are found by solving simultaneously the equations of the two loci. The geometric problem of finding the intersection of two straight lines is thus equivalent to the algebraic problem of solving a simultaneous pair of linear equations. To find the points of intersection of a straight line, say, $y = x + 1$ and the parabola $y = 2x^2$, one must solve simultaneously the pair of equations

$$y = x + 1.$$
$$y = 2x^2.$$

Subtracting,

$$2x^2 - x - 1 = 0,$$

the roots of which are $x = 1, x = -\frac{1}{2}$. When $x = 1$, it is found, from either of the original equations, that $y = 2$. Thus, one point of intersection is the point (1, 2). Similarly, the other point of intersection is the point $(-\frac{1}{2}, \frac{1}{2})$.

Exercises

Find the point or points of intersection of the following pairs of loci:

1. $y = x^2$ and $y = x$.
2. $y = 2x^2$ and $y = \frac{1}{2}x + 1$.
3. $y = x^3$ and $y = 2x$.
4. $y = -x$ and $x^2 + y^2 = 25$.
5. $y^2 = 4x$ and $y = 2x$.
6. $y^2 = 9x$ and $y = x - 1$.
7. $y = x^{3/2}$ and $x = 4$.
8. $y = x^{3/2}$ and $y = x$.

Miscellaneous Exercises

1. Find the slope, the Y-intercept, and the X-intercept for the following:

(*a*) $y = 2x - 3$; (*b*) $y = x + 2$; (*c*) $3y - 6x = 10$.

2. Write the equations of the lines determined by the following data:

(*a*) Slope 2 Y-intercept 5.
(*b*) Slope −2 Y-intercept 5.
(*c*) Slope 2 Y-intercept −5.
(*d*) Slope −2 Y-intercept −5.
(*e*) Slope −2 X-intercept 4.

3. Does the line $3y - 2x = 1$ pass through the point:

(*a*) (1, 1). (*c*) (−2, −1). (*e*) (3, 4).
(*b*) (2, 2). (*d*) (0, 0).

4. Find the equation of a straight line with slope 2 and passing through the point (3, 2).

5. Write the equations of the lines determined by the following data:

(*a*) Slope 1 and passing through (1, 1).
(*b*) Slope −1 and passing through (−1, 1).
(*c*) Slope 2 and passing through (1, −3).
(*d*) Slope −3 and passing through (−2, −1).

6. Write the equation of a line passing through (2, 1) and (3, −5).

7. Write the equations of the lines passing through the following pairs of points:

(*a*) (1, 1) and (2, 3). (*d*) (1, −5) and (−2, −3).
(*b*) (3, −1) and (−2, 1). (*e*) (0, 2) and (3, 0).
(*c*) (2, −3) and (2, 1). (*f*) (0, 0) and (−3, 2).

8. Make two suitable graphs upon a single sheet of squared paper from the following data, giving the highest and the lowest average closing price of twenty-five leading stocks listed on the New York Stock Exchange for the years given in the table:

Year	Highest	Lowest
1913	94.56	79.58
1912	101.40	91.41
1911	101.76	86.29
1910	111.12	86.32
1909	112.76	93.24
1908	99.04	67.87
1907	109.88	65.04
1906	113.82	93.36
1905	109.05	90.87
1904	97.73	70.66
1903	98.16	68.41
1902	101.88	87.30

Should smooth curves be drawn through the points plotted from this table?

9. Plot the data given in following table upon squared paper. Use the same horizontal axis for all three curves. Put the temperature curves above the discharge curve, using the same horizontal (time)

scale for both. Let 1 cm. on the vertical scale represent 0.1 sec.-ft.[1] discharge, and 10° temperature. Start the temperature scale with 60°, and place the 60, 6 cm. above the horizontal scale. Start the discharge scale with 3.4 placed on the horizontal scale.

DISCHARGE OF A SEEPAGE DITCH

Time, Aug. 24, 1905	Discharge of ditch, sec.-ft.	Temp. of water, °F.	Temp. of air, °F.
8:00 a. m.	3.72	65	...
9:00 a. m.	3.70	70	74
11:00 a. m.	3.66	79	84
1:30 p. m.	3.52	83	85
2:15 p. m.	3.49	...	...
3:30 p. m.	3.52	84	90
5:30 p. m.	3.66	78	88
6:00 p. m.	3.73	...	...
8:00 p. m.	3.84	67	76

10. Plot data given in the following table.[2] Plot d/D along the horizontal axis, using 1 cm. to represent $\frac{1}{10}$ unit. Plot velocities along the vertical axis, using 1 cm. to represent 0.02 ft. per second. Sketch a smooth curve among the points. The curve may not pass through

RELATION BETWEEN VELOCITY AND DEPTH AT A POINT IN THE LOWER MISSISSIPPI

Depth at observed point = d; whole depth = D

$\frac{d}{D}$	Velocity, feet per second	$\frac{d}{D}$	Velocity, feet per second
0.0	3.195	0.5	3.228
0.1	3.230	0.6	3.181
0.2	3.253	0.7	3.127
0.3	3.261	0.8	3.059
0.4	3.252	0.9	2.976

[1] Second-foot (or sec.-ft.), when applied to the measurement of flow of water means 1 cu. ft. per second.

[2] The velocity at any point of a moving stream is determined by a current meter placed at that point.

all the plotted points.[1] Begin to number the vertical axis with 2.90. The true origin will be far below the sheet of paper.

11. Write the equations of the following curves after being translated: 2 units to the right; 3 units to the left; 5 units up; 1 unit down; 2 units to the left and 1 unit down:

(*a*) $y = 2x^2$. (*d*) $y = \frac{1}{x}$. (*g*) $y = x^{3/2}$.

(*b*) $y = -3x^2$. (*e*) $y = \frac{1}{x^2}$. (*h*) $y = x^{3/3}$.

(*c*) $y = x^3$. (*f*) $y = \frac{1}{x^{1/2}}$.

Sketch each curve in its original and also in its translated position.

12. Write the equation of each curve of Exercise 11 when reflected in the X-axis; in the Y-axis; in the line $y = x$. Sketch each curve before and after reflection.

13. Shear each curve of Exercise 11 in the line $y = \frac{1}{2}x$; in the line $y = -\frac{1}{2}x$; in the line $y = x$; in the line $y = -x$. Sketch each curve in its original and sheared position.

14. Draw on a sheet of coordinate paper the lines $x = 0$, $x = 1$, $x = -1$, $y = 0$, $y = 1$, $y = -1$. Shade the regions in which the hyperbolic curves lie with vertical strokes; and those in which the parabolic curves lie with horizontal strokes. *Write down* all that the resulting figure tells you.

15. Consider the following: $y = x^2$, $y = x^{-3}$, $y = x^{4/3}$, $xy = -1$, $y = -x^3$, $y^3 = x^4$, $y^4 = x^2$, $xy = 1$, $x^3 = -y^2$, $x^4 = -y^2$. In which equation is y an increasing function of x in the first quadrant? For which does the slope of the curve increase in the first quadrant? For which does the slope of the curve decrease in the first quadrant?

16. Which of the curves of Exercise 11 pass through (0, 0)? Through (1, 1)? Through (−1, −1)?

17. Find the vertex of the curve $y = x^2 - 24x + 150$.

NOTE: The lowest point of the parabola $y = x^2$ may be called the *vertex*.

SUGGESTION: It is necessary to put the equation in the form $y - b = (x - a)^2$. This can be done as follows: Add and subtract 144 on the right side of the equation, obtaining

$$y = x^2 - 24x + 144 - 144 + 150,$$

[1] This curve is called a vertical velocity curve. In practical work, however, velocities are plotted along the horizontal axis and depths along the vertical axis, and down from the origin. Your drawing gives the usual form of plotting if it is turned 90° in a clockwise direction. Vertical velocity curves are parabolic in shape, with the axis of the parabola parallel to the surface of the water.

$$y = (x - 12)^2 + 6,$$

or

$$y - 6 = (x - 12)^2$$

Then this is the curve $y = x^2$ translated 12 units to the right and 6 units up. Since the vertex of $y = x^2$ is at the origin, the vertex of the given curve must be at the point (12, 6).

18. Find the vertex of the parabola $y = x^2 - 6x + 11$.

19. Find the vertex of $y = x^2 + 8x + 1$.

20. Find the vertex of $4 + y = x^2 - 7x$.

21. Find the vertex of $y = 9x^2 + 18x + 1$.

22. Does the line $2y = 5x - 1$ pass through the point of intersection of $y = 2x + 1$ and $y = 3x - 2$?

23. At what point does the line $y = 3x - 2$ intersect a line which passes through the point (1, 2) with slope -1?

24. A line passes through the point $(-1, 3)$ and the point $(3, -2)$. Where does this line intersect the line $y = x - 5$?

25. What are the points of intersection of the line $y = mx + 1$ and the curve $y^2 = 4x$? For what value of m do these two points coincide? Interpret geometrically.

CHAPTER III

THE CIRCLE AND THE CIRCULAR FUNCTIONS

42. Equation of the Circle.—In rectangular coordinates the abscissa x and the ordinate y of any point P (as OD and DP, Fig. 46) form two sides of a right triangle whose hypotenuse squared is $x^2 + y^2$. If the point P move in such manner that the length of this hypotenuse remains fixed, the point P describes a circle whose center is the origin (Fig. 46). The equation of this circle is, therefore,

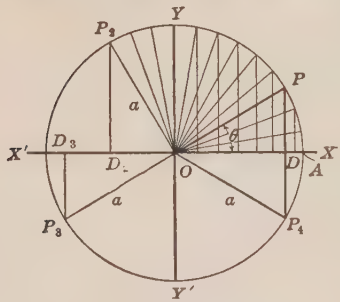


FIG. 46.—The circle.

$$\mathbf{x^2 + y^2 = a^2}, \qquad (1)$$

where $a = OP$, the radius of the circle.

It is sometimes convenient to write the equation of the circle, solved for y, in the form

$$\mathbf{y = \pm \sqrt{a^2 - x^2}}. \qquad (2)$$

This gives, for each value of x, the two corresponding equal and opposite ordinates.

To translate the circle of radius a so that its center shall be at the point (h, k), it is merely necessary to write

$$\mathbf{(x - h)^2 + (y - k)^2 = a^2}. \qquad (3)$$

This is the general equation of any circle in the plane XY, for it locates the center at any desired point (h, k), and provides for any desired radius a.

Exercises

1. Write the equations of the circles with center at the origin having radii 3, 4, 11, $\sqrt{2}$ respectively.

2. Write the equation of each circle described in Exercise 1 in the form $y = \pm\sqrt{a^2 - x^2}$.

3. Which of the following points lie on the circle $x^2 + y^2 = 169$: (5, 12), (0, 13), (−12, 5), (10, 8), (9, 9), (9, 10)?

4. Which of the following points lie inside and which lie outside of the circle $x^2 + y^2 = 100$: (7, 7), (10, 0), (7, 8), (6, 8), (−5, 9), (−7, −8), (2, 3), (10, 5), $(\sqrt{40}, \sqrt{50})$, $(\sqrt{19}, 9)$?

43. The equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{1}$$

may be put in the form of Eq. (3), Sec. **42,** for it may be written

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$

or

$$(x + g)^2 + (y + f)^2 = (\sqrt{g^2 + f^2 - c})^2, \tag{2}$$

which represents a circle of radius $\sqrt{g^2 + f^2 - c}$ whose center is at the point $(-g, -f)$. In case $g^2 + f^2 - c < 0$, the radical becomes imaginary, and the locus is not a real circle; that is, coordinates of no points in the plane XY satisfy the equation. If the radical be zero, the locus is a single point.

44. *Any equation of the second degree, in two variables, lacking the term xy and having like coefficients in the terms x^2 and y^2, represents a circle, real, null, or imaginary.* The general equation of the second degree in two variables may be written

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, \tag{3}$$

for, when only two variables are present, there can be present three terms of the second degree, two terms of the first degree, and one term of the zeroth degree. When $a = b$ and $h = 0$ this reduces to the form of Eq. (1) above after dividing through by a.

Exercises

Find the centers and the radii of the circles given by the following equations:

1. $x^2 + y^2 = 25$. Also determine which of the following points are on this circle: (3, 4), (5, 5), (4, 3), (−3, −4), (−3, 4), (5, 0), $(2, \sqrt{21})$.

2. $x^2 + y^2 = 16$.

3. $x^2 + y^2 - 4 = 0$.

4. $x^2 + y^2 - 36 = 0$.

5. $x^2 + y^2 + 2x = 0$.

6. $y = \pm\sqrt{169 - x^2}$. Also find the slope of the diameter through the point (5, 12). Find the slope of the tangent at (5, 12).

7. $9 - x^2 - y^2 = 0$.
8. $x^2 + y^2 - 6y = 16$.
9. $x^2 - 2x + y^2 - 6y = 15$.
10. $(x + a)^2 + (y - b)^2 = 50$.
11. $x^2 + y^2 + 6x - 2y = 10$.
12. $x^2 + y^2 - 4x + 6y = 12$.
13. $x^2 + y^2 - 4x - 8y + 4 = 0$.
14. $3x^2 + 3y^2 + 6x + 12y - 60 = 0$.
15. Is $x^2 + 2y^2 + 3x - 4y - 12 = 0$ the equation of a circle? Why?
16. Is $2x^2 + 2y^2 - 3x + 4y - 8 = 0$ the equation of a circle? Why?

45. Angular Magnitude.—By the **magnitude of an angle** is meant the amount of rotation of a line about a fixed point. If a line OA rotate in the plane XY about the fixed point O to the position OP, the line OA is called the **initial side** and the line OP is called the **terminal side** of the angle AOP. The notion of angular

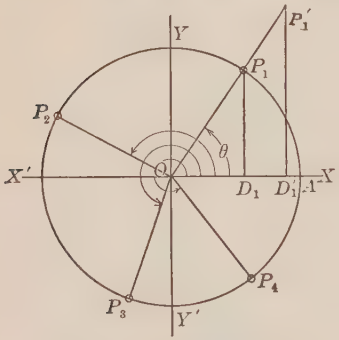


FIG. 47.—Positive angles.

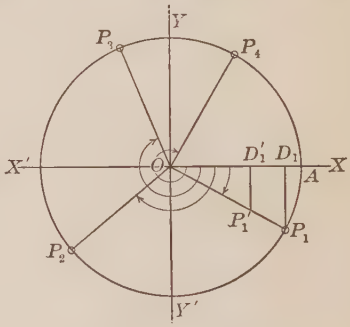


FIG. 48.—Negative angles.

magnitude as introduced in this definition is more general than that used in elementary geometry. There are two new and very important consequences that follow therefrom:

1. Angular magnitude is unlimited in respect to size—that is, it may be of any amount whatsoever. An angular magnitude of 100 right angles or twenty-five complete rotations is quite as possible, under the present definition, as an angle of smaller amount.

2. Angular magnitude exists, under the definition, in two opposite senses—for rotation may be **clockwise** or **anti-clockwise.**

As is usual in mathematics, the two opposite senses are distinguished by the terms **positive** and **negative.** If the rotation be anti-clockwise, the angle is positive; if clockwise, it is negative. In Fig. 47, AOP_1, AOP_2, AOP_3, and AOP_4 are positive angles. In designating an angle its initial side is always named first. Thus in Fig. 47, AOP_1 designates a positive angle of initial side OA. In Fig. 48, AOP_1, AOP_2, AOP_3, and AOP_4 are negative angles.

In Cartesian coordinates, OX is usually taken as the initial line for the generation of angles. If the terminal side of an angle falls within the first quadrant, the angle is said to be an angle of the first quadrant. If the terminal side of any angle falls within the second quadrant, it is said to be an angle of the second quadrant, etc.

Two angles which differ by any multiple of 360° are called **congruent angles.** We shall find that in *certain cases* congruent angles may be substituted for each other without modifying results.

The theorem in elementary geometry, that angles at the center of a circle are proportional to the intercepted arcs, holds, obviously, for the more general notion of angular magnitude here introduced.

46. Units of Measure.—Angular magnitude, like all other magnitudes, must be measured by the application of a suitable unit of measure. Four systems are in common use:

1. *Right-angle System.*—Here the unit of measure is the right angle, and all angles are given by the number of right angles and fraction of a right angle therein contained. This unit is familiar to the student from elementary geometry. A practical illustration is the scale of a mariner's compass, in which the right angles are divided into halves, quarters, and eighths.

2. *The Degree System.*—Here the unit is the angle corresponding to one three-hundred-sixtieth of a complete rotation. This system, with the sexagesimal subdivisions (division by sixtieths) into **minutes** and **seconds,** is familiar to the student. This system dates back to remote antiquity. It was used by, if it did not originate among, the Babylonians.

3. *The Hour System.*—In astronomy, the angular magnitude about a point is divided into 24 hours, and these into minutes

and seconds. This system is analogous to our system of measuring time.

4. *The Radian or Circular System.*—If a circular arc be drawn with center at the vertex of an angle, as in Fig. 49, the length of the arc $s = AP$ is a function of the angle $\theta = AOP$ and of the radius $r = OA$. For example, if the angle θ be measured in the right-angle system, the relation between s, r, and θ is given by the equation

$$s = \frac{\pi}{2}\theta r.$$

In mechanics, mathematical physics, and pure mathematics it is found convenient to choose a unit for measuring angles, so that

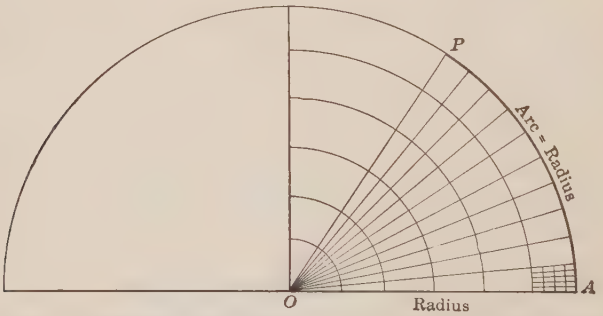


FIG. 49.—Definition of *the Radian.* The angle AOP is *one Radian.*

this relation between s, r, and θ takes the simplest form possible. This is accomplished by choosing, as a unit, the angle which is subtended, at O, by an arc s whose length is equal to that of the radius r. This unit of angular measure is called a **radian.** Its size is shown in Fig. 49. In the case of an angle of 2 radians, the arc is twice the radius, or arc/radius $= 2$; whereas if the arc is half the radius so that arc/radius $= \frac{1}{2}$, the angle is $\frac{1}{2}$ radian. In general, the measure of an angle in radians is equal to the arc length divided by the radius of that arc; *i.e.*,

$$\theta = \frac{s}{r} \tag{1}$$

or

$$s = r\theta. \tag{2}$$

This last equation expresses the simple relation, referred to above, between s, r, and θ which holds when θ is measured in radian measure, and it is the absence of a numerical factor from this equation which makes the radian such a useful unit.

Inasmuch as the radius is contained 2π times in a circumference, we have the equivalents:

$$2\pi \text{ radians} = 360°.$$

or

$$1 \text{ radian} = 57° 17'44''.8 = 57°17'.7 = 57°.3 \text{ nearly.}$$
$$1° = 0.01745 \text{ radians.}$$

The following equivalents are of special importance:

$$\text{a straight angle} = \pi \text{ radians.}$$

$$\text{a right angle} = \frac{\pi}{2} \text{ radians.}$$

$$60° = \frac{\pi}{3} \text{ radians.}$$

$$45° = \frac{\pi}{4} \text{ radians.}$$

$$30° = \frac{\pi}{6} \text{ radians.}$$

There is no generally adopted scheme for writing angular magnitude in radian measure. We shall use the superior Roman letter "r" to indicate the measure. For example, $18° = 0.31416^{r}$.

Since the circumference of a circle is incommensurable with its diameter, it follows that the number of radians in an angle is always incommensurable with the number of degrees in the angle.

The speed, or angular velocity, of rotating parts is usually given either in revolutions per minute (abbreviated "r.p.m.") or in radians per second.

The expression for the area A of a circular sector of central angle θ and radius r also takes particularly simple form when θ is measured in radians. In fact, since this area is then the $\theta/2\pi$th part of the whole area of the circle, it is given by

$$A = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}. \tag{3}$$

47. Uniform Circular Motion.—Suppose the line OP (Fig. 46) is revolving counter-clockwise at the rate of k^{r} per second, the

angle AOP in radians is then kt, t being the time in seconds required for OP to turn from the initial position OA. If we call θ the angle AOP, we have $\theta = kt$ as the equation defining the motion. The following terms are in common use:

1. The **angular velocity** of the uniform circular motion is k (radians per second).

2. The **amplitude** of the uniform circular motion is a, the radius of the circle.

3. The **period** of the uniform circular motion is the number of seconds required for one revolution.

4. The **frequency** of the uniform circular motion is the number of revolutions per second.

Sometimes the unit of time is taken as 1 min. Also the motion is sometimes clockwise, or negative.

Exercises

1. Express each of the following in radians: 135°, 330°, 225°, 15°, 150°, 75°, 120°. (Do *not* work out in decimals; use π.)

2. Express each of the following in degrees: 0.2^r, $\frac{1}{5}\pi^r$, $\frac{3}{2}\pi^r$, $\frac{5}{4}\pi^r$.

3. How many revolutions per minute is 20 radians per second?

4. The angular velocity, in radians per second, of a 36-in. automobile tire is required, when the car is making 20 miles per hour.

5. What is the angular velocity in radians per second of a 6-ft. drive wheel, when the speed of the locomotive is 50 miles per hour?

6. The frequency of a cream separator is 6800 r.p.m. What is its period, and its angular velocity in radians per second?

7. A wheel is revolving uniformly 30^r per second. What is its period and frequency?

8. The speed of the turbine wheel of a 5-hp. DeLaval steam turbine is 30,000 r.p.m. What is the angular velocity in radians per second?

9. In the expression, given in Sec. **46**, for the area of a circular sector, replace θ by its value in terms of s and r. In what way does the resulting formula remind you of a formula for the area of a triangle? This similarity should aid you in remembering the equation for A.

10. An angle of 2 radians is inscribed with its vertex at the center of a circle of radius 4 in. What is the length of the intercepted arc?

11. What is the area of a circular sector of radius 5 in. and central angle 0.3 radians?

12. Find the radius of a circle such that the length of the arc of 1° shall be 1 in. Here the length of the circumference is, obviously, 360 in. The radius is then $360/2\pi$, or approximately 57.3 in.

13. Find the radius of a circle such that 10° of arc shall measure $\frac{1}{4}$ in. Here 40° must measure 1 in.; therefore (see Problem 12) the radius must be 57.3/40, or approximately 1.432 in.

14. Find the radius of a circle such that 10° of arc shall measure $\frac{1}{5}$ in. Here 50° must measure 1 in.; therefore (see Problem 12) the radius must be 57.3/50, or approximately 1.15 in.

15. What must the radius of a circle be in order that a central angle of 3 radians intercept an arc of length 1 in.?

16. If the radius of a circle be 1 in., what is the length of arc intercepted by a central angle of $\pi/18$ radians. What is the value of this angle in degrees?

48. The Circular, or Trigonometric Functions.—To each point on the circle $x^2 + y^2 = a^2$ there corresponds not only an abscissa and an ordinate, but also an angle $\theta < 360°$, as shown in Fig. 46. This angle is called the **direction angle,** or **vectorial angle,** of the point P. When θ is given, x, y, and a are not determined, but the *ratios* y/a, x/a, y/x, and their reciprocals, a/y, a/x, x/y *are determined.* Hence **these ratios are, by definition** (Sec. 3), **functions of** θ. They are known as the **circular,** or **trigonometric functions** of θ, and are named and written as follows:

Function of θ	Name	Written
y/a	**sine of** θ	sin θ
x/a	**cosine of** θ	cos θ
y/x	**tangent of** θ	tan θ
x/y	**cotangent of** θ	cot θ
a/x	**secant of** θ	sec θ
a/y	**cosecant of** θ	csc θ

The circular functions are usually thought of in the above order; that is, in such order that the first and last, the middle two, and those intermediate to these are reciprocals of each other.

The names of the six ratios must be committed to memory. They should be committed, using the *names* of x, y, and a as follows:

Ratio	Written
ordinate/radius	**sin** θ
abscissa/radius	**cos** θ
ordinate/abscissa	**tan** θ
abscissa/ordinate	**cot** θ
radius/abscissa	**sec** θ
radius/ordinate	**csc** θ

The right triangle DOP (Fig. 46) of sides x, y and a, whose ratios give the functions of the angle XOP, is often called the **triangle of reference** for this angle. It is obvious that the *size* of the triangle of reference has no effect of itself upon the value of the functions of the angle. Thus, in Fig. 47 either D_1OP_1 or $D_1'OP_1'$ may be taken as the triangle of reference for the angle θ. Since the triangles are similar, we have

$$\frac{P_1D_1}{OD_1} = \frac{P_1'D_1'}{OD_1'}, \quad \frac{P_1D_1}{OP_1} = \frac{P_1'D_1'}{OP_1'},$$

etc., which shows that identical ratios or trigonometric functions of θ are derived from the two triangles of reference.

49. Signs of the Functions.—The circular functions have, of course, the algebraic signs of the ratios that define them. Of the three numbers entering these ratios, the distance, or radius a, is always to be taken as positive. It enters the ratios, therefore, always as a positive number. The abscissa and the ordinate, x and y, have the algebraic signs appropriate to the quadrants in which P falls. The student should determine the signs of the functions in each quadrant, as follows (see Fig. 46):

	First quadrant	Second quadrant	Third quadrant	Fourth quadrant
Sine.....	+	+	−	−
Cosine.....	+	−	−	+
Tangent.....	+	−	+	−

Of course, the reciprocals have the same signs as the original functions.

The signs are readily remembered by the following scheme:

Sine		Cosine		Tangent	
+	+	−	+	−	+
−	−	−	+	+	−
Cosecant		Secant		Cotangent	

50. Graphic Computation of Circular Functions.—Approximate determination of the numerical values of the circular functions of any given angle may be made graphically on ordinary coordinate paper. Locate the vertex of the angle at the intersection of any two lines of the squared paper (form $M1$). Let the initial side of the angle coincide with one of the rulings of the squared paper and lay off the terminal side of the angle by means of a protractor. If the sine or cosine is desired, describe a circle about the vertex of the angle as center using a radius appropriate to the scale of the squared paper—for example, a radius of 10 cm. on coordinate paper ruled in centimeters and fifths (form $M1$) permits direct reading to one-fiftieth of the radius a and, by interpolation, to one-hundredth of the radius a. The ordinate and the abscissa of the point of intersection of the terminal side of the angle and the circle may then be read and the numerical value of sine and cosine computed by dividing each of these by the length of the radius.

If the numerical value of the tangent or cotangent be required, the construction of a circle is not necessary. The angle should be laid off as above described, and a triangle of reference constructed. To avoid long division, the abscissa of the triangle of reference may be taken equal to 50 or 100 mm. for the determination of the tangent; and the ordinate may be taken equal to 50 or 100 mm. for the determination of the cotangent.

The following table (Table IV) contains the trigonometric functions of acute angles for increments of 10° of the argument from 0 to 90° (see the end of the book for a larger table):

TABLE IV

Natural Trigonometric Functions to Two Decimal Places

θ°	θ^r	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0	0.00	0.00	1.00	0.00	∞	1.00	∞
10	0.17	0.17	0.98	0.18	5.67	1.02	5.76
20	0.35	0.34	0.94	0.36	2.75	1.06	2.92
30	0.52	0.50	0.87	0.58	1.73	1.15	2.00
40	0.70	0.64	0.77	0.84	1.19	1.31	1.56
50	0.87	0.77	0.64	1.19	0.84	1.56	1.31
60	1.05	0.87	0.50	1.73	0.58	2.00	1.15
70	1.22	0.94	0.34	2.75	0.36	2.92	1.06
80	1.40	0.98	0.17	5.67	0.18	5.76	1.02
90	1.57	1.00	0.00	∞	0.00	∞	1.00

Exercises

1. Lay off angles of 10°, 20°, and 30° with a protractor and find by graphic construction the sine of each angle.

2. Lay off angles of 10°, 20°, and 30°, with a protractor, and find by graphic construction the cosine of each angle. Record the results of this and the previous exercise in a suitable table.

3. Construct an angle of 45°, and determine graphically its tangent.

4. Find the value of tan 60° by graphic construction.

5. Find by graphic construction all the functions of 15°.

NOTE: A protractor is not needed, as angles of 45° and 30° may be constructed with ruler and compass.

6. Find the sine, cosine, and tangent of 75°.

7. Which is greater, sec 40° or csc 50°?

8. Determine the angle whose tangent is $\frac{1}{2}$.

9. Find the angle whose sine is 0.6.

10. Which is greater, sin 40° or 2 sin 20°?

11. Does an angle exist whose tangent is 1,000,000?

51. The Functions of 30°, 60°, and 45°.—Elaborate means for computing the six functions have been devised, and the values of the functions have been placed in convenient tables for use. The functions are usually printed to three, four, five, or six decimal places, but tables of eight, ten, and fourteen places exist. The

functions of only a few angles can be computed by elementary means. These angles are, however, especially important.

1. *The Functions of* 30°.—It has been observed above that the size of the triangle of reference has no effect upon the values of the functions of an angle. It is, therefore, sensible to choose a triangle of the most convenient size. In Fig. 50*a*, angle BOA is 30°, and $BB'O$ is an equilateral triangle. Thus, if the radius $a = OB$ be 2 units long, $AB = y = 1$, and. from the relation

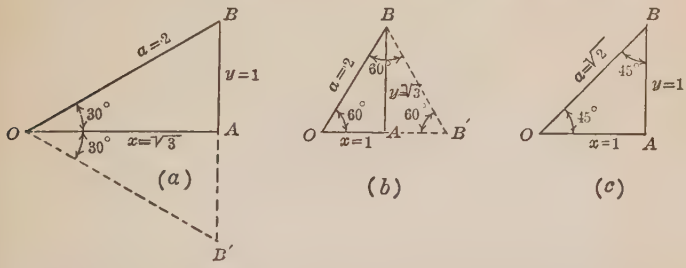


FIG. 50.—Triangles of reference for angles of 30°, 60°, and 45°.

$$OA = \sqrt{\overline{OB}^2 - \overline{BA}^2},$$
$$OA = x = \sqrt{3}.$$

From this reference triangle the following ratios may then be read:

$$\sin 30° = \frac{1}{2}. \qquad \cot 30° = \sqrt{3}.$$
$$\cos 30° = \frac{\sqrt{3}}{2}. \qquad \sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$
$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \qquad \csc 30° = 2.$$

2. *The Functions* of 60°.—In a similar manner, the six functions of 60° can be read from the triangle shown in Fig. 50*b*. The results are:

$$\sin 60° = \frac{\sqrt{3}}{2}. \qquad \cot 60° = \frac{\sqrt{3}}{3}.$$
$$\cos 60° = \frac{1}{2}. \qquad \sec 60° = 2.$$
$$\tan 60° = \sqrt{3}. \qquad \csc 60° = \frac{2\sqrt{3}}{3}.$$

3. *The Functions of* 45°.—A convenient triangle of reference from which to determine the functions of 45° is shown in Fig. 50c. From this triangle it follows that:

$$\sin 45° = \frac{\sqrt{2}}{2}. \qquad \cot 45° = 1.$$

$$\cos 45° = \frac{\sqrt{2}}{2}. \qquad \sec 45° = \sqrt{2}.$$

$$\tan 45° = 1. \qquad \csc 45° = \sqrt{2}.$$

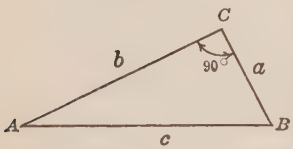


FIG. 51.—Diagram for Sec. **52.**

52. The Defining Equations Cleared of Fractions.—The student should learn to read off the circular functions of any angle of a right triangle, no matter in what position the triangle may lie. Thus, from Fig. 51 he may write:

$$\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypothenuse}}. \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}}.$$

$$\cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypothenuse}}. \qquad \sec A = \frac{\text{hypothenuse}}{\text{adjacent}}.$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent side}}. \qquad \csc A = \frac{\text{hypothenuse}}{\text{opposite}}.$$

These are easy to learn from the definitions "ordinate/radius," "abscissa/radius," etc., by changing "radius" to "hypothenuse," and noticing that "opposite" and "ordinate" and that "adjacent" and "abscissa" begin with the same letters.

Even more important are the equations defining the trigonometric functions *when cleared of fractions.* In this form the equations are quite as useful as the original ratios. They are written:

$$\mathbf{y = a \sin \theta.} \qquad \mathbf{y = x \tan \theta.} \qquad \mathbf{a = x \sec \theta.}$$

$$\mathbf{x = a \cos \theta.} \qquad \mathbf{x = y \cot \theta.} \qquad \mathbf{a = y \csc \theta.}$$

As applied to the right-angled triangle (see Fig. 51) these three sets of equations may be stated in words as follows:

I. *Either leg of a right triangle is equal to the hypotenuse multiplied by the sine of the opposite, or by the cosine of the adjacent, angle.*

II. *Either leg of a right triangle is equal to the other leg multiplied by the tangent of the opposite, or by the cotangent of the adjacent, angle.*

III. *The hypotenuse of a right triangle is equal to either leg multiplied by the secant of the angle adjacent, or by the cosecant of the angle opposite that leg.*

These statements should be committed to memory.

53. Orthographic Projection.—In elementary geometry we learned that the projection of a given point P upon a given line or plane is the foot of the perpendicular dropped from the given point upon the given line or plane. Likewise, if perpendiculars be dropped from the end points A and B of any line segment AB upon a given line or plane, and if the feet of these perpendiculars be called P and Q, respectively, then the line segment PQ is called the **projection** of the line AB. Also, if perpendiculars be dropped from all points of a given curve AB upon a given plane MO, the locus formed by the feet of all perpendiculars so drawn is called the **projection of the given curve upon the plane MO.**

To emphasize the fact that the projections were made by using perpendiculars to the given plane, it is customary to speak of them as **orthogonal** or **orthographic** projections.

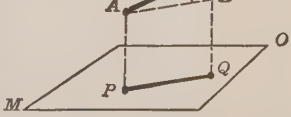


FIG. 52.—Orthographic projection.

The length of the orthographic projection of a line segment AB upon a plane is obtained by multiplying the length AB by the cosine of the angle θ between AB and the plane (see Fig. 52) for, by the rule of the previous section, $PQ = AB'$ is obtained by multiplying the hypothenuse AB by the cosine of the adjacent angle θ. In just the same way, when two lines are in a plane, the projection on one of a segment of the other is equal to the length of this segment times the cosine of the angle between the two lines. This use of the cosine as the "projecting" function is of fundamental importance, and is one with which the student should make himself entirely familiar.

Exercises—Projections

Make a sketch, preferably to scale, for each exercise.

1. A stretched guy rope 75 ft. long makes an angle of 60° with the horizontal. What is the length of the projection of the rope on a horizontal plane? What is the length of the projection of the rope on a vertical plane?

2. Find the lengths of the projections of the line through the origin and the point $(1, \sqrt{3})$ upon the OX- and OY-axes, if the line be 12 in. long.

3. A line 8 in. long makes an angle of 45° with the X-axis. What is the length of its projection on the X-axis?

4. A velocity of 20 ft. per second is represented as the diagonal of a rectangle the longer side of which makes an angle of 30° with the diagonal. Find the components of the velocity along each side of the rectangle.

5. Show that the projections of a fixed line OA upon all other lines drawn through the point O are chords of a circle of diameter OA.

6. Find the projection of the side of a regular hexagon upon the three diagonals passing through one end of the given side, if the side of the hexagon be 20 ft.

Exercises—Solution of Right Triangles

Sketch the triangle in each case. It is preferable to letter the triangles uniformly, c hypothenuse, a and b legs, and A, B, C the angles opposite the sides of the same name. Computed results should be *checked* by drawing a diagram to scale.

1. Find the legs of a right triangle if the hypothenuse be 5280 ft., and the acute angles 30° and 60°.

Suggestion: By Sec. **52,** Par. I, the leg opposite the angle 30°, is 5280 sin 30°; the leg adjacent to the angle 30° is 5280 cos 30°. Substitute the values of sin 30° and cos 30° and compute.

2. In a right triangle ABC, $A = 20°$, $a = 20$ ft. Find the hypothenuse c and the leg b.

Suggestion: Use Par. III, Sec. **52,** and Table IV.

3. In a right triangle ABC, $A = 60°$, $a = 20$ ft. Find the other leg b and the hypothenuse c, and check results by substituting in the formula $(c - a)(c + a) = b^2$. Use Par. II, Sec. **52.**

4. Solve the right triangle whose hypothenuse is 120 ft. and in which $A = 45°$.

5. Find the shorter leg of a right triangle if the hypothenuse $c = 111$ ft. and $A = 60°$.

54. Functions of 120°, 135°, 150°, etc.—For angles greater than 90° the functions must be obtained from the original definition given, in Sec. **48**, in terms of ordinate, abscissa, and radius. There are triangles of reference, however, corresponding to the angles 120°, 135°, 150°, 210°, 225°, 240°, 300°, 315°, and 330°, which are geometrically similar to the triangles in Fig. 50. For instance, in the case of 120° the reference triangle *ODP* in Fig. 53 is similar to triangle *BAO* of Fig. 51*a*, and the sides of the triangle *ODP* can be labeled by referring to Fig. 51, taking precaution to remember that a point in the second quadrant has a negative abscissa. From Fig. 53, sin 120° = $\sqrt{3}/2$, while cos 120° = $-\frac{1}{2}$ and tan 120° = $-\sqrt{3}$.

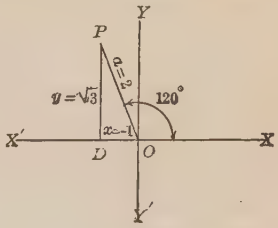


FIG. 53.—Reference triangle for 120°.

Exercises

1. The student will fill in the blanks in the following table with the correct numerical value and the correct sign of each function:

Function	120°	135°	150°	210°	225°	240°	300°	315°	330°
Sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$
Cos	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
Tan	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$
Cot	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$
Sec	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$
Csc	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2

2. Write down the functions of 390° and 405°.

3. The tangent of an angle is 1. What angle $<360°$ may it be?

4. Cos $\theta = -\frac{1}{2}$. What two angles $<360°$ satisfy the equation?

5. Sec $\theta = 2$. Solve for all angles $<360°$.

6. Csc $\theta = -\sqrt{2}$. Solve for $\theta < 360°$.

55. Functions of 0° and 90°.—In Fig. 46, let the angle AOP decrease toward zero, the point P remaining on the circumference of radius a. Then y or PD decreases toward zero. Therefore, $\sin 0° = 0$. Also x or OD increases toward the value of a so that the ratio x/a becomes unity, or $\cos 0° = 1$. Likewise, the ratio y/x becomes zero, or $\tan 0° = 0$.

The reciprocals of these functions change as follows: As the angle AOP approaches zero, the ratio a/y **increases in value without limit,** *or* **the cosecant becomes infinite.** In symbols (see Sec. **25**) $\csc 0° = \infty$, Likewise, $\cot 0° = \infty$, but $\sec 0° = 1$.

In a similar way the functions of 90° may be investigated. The following table gives the variation of the functions as the angle varies from 0° to 90°, from 90° to 180° etc.:

Angle	From 0° to 90°	From 90° to 180°	From 180° to 270°	From 270° to 360°
Sin.....	0 to + 1	+ 1 to 0	0 to − 1	− 1 to 0
Cos.....	+ 1 to 0	0 to − 1	− 1 to 0	0 to 1
Tan.....	0 to + ∞	− ∞ to 0	0 to ∞	− ∞ to 0
Cot.....	+ ∞ to 0	0 to − ∞	∞ to 0	
Sec.....	+ 1 to + ∞	− ∞ to − 1	− 1 to − ∞	
Csc.....	+ ∞ to + 1	+ 1 to + ∞	− ∞ to − 1	

The student is to supply the results for the last two columns.

56. Fundamental Relations.—The trigonometric functions are not independent of each other. Because of the relation $x^2 + y^2 = a^2$, it is possible to compute the numerical or absolute values of the remaining five functions when the value of any one of the six is given. This may be accomplished by means of the fundamental formulas derived below.

Divide the members of the equation

$$x^2 + y^2 = a^2 \qquad (1)$$

by a^2. Then

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1,$$

or,

$$\sin^2 \theta + \cos^2 \theta = 1. \qquad (2)$$

Likewise, divide Eq. (1) through by x^2; then

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{a}{x}\right)^2,$$

or

$$\sec^2 \theta = 1 + \tan^2 \theta. \tag{3}$$

Also divide Eq. (1) through by y^2; then

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{a}{y}\right)^2,$$

or

$$\csc^2 \theta = 1 + \cot^2 \theta. \tag{4}$$

Also, since

$$\frac{y/a}{x/a} = \frac{y}{x},$$

we obtain

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \tag{5}$$

and likewise

$$\cot \theta = \frac{\cos \theta}{\sin \theta}. \tag{6}$$

Equations (2) to (6) are the fundamental relations between the six trigonometric functions. They must be committed to memory by the student.

Exercises

All angles in the following exercises are supposed to be less than 90°.

1. Sin $\theta = \frac{1}{5}$. Find the values for the other five circular functions.

Draw a right triangle whose hypotenuse is 5, whose altitude is 1, and whose base coincides with OX. In other words, make $a = 5$ and $y = 1$ in Fig. 54. Calculate $x = \sqrt{25 - 1} = 2\sqrt{6}$ and write down all of the functions from their definitions.

2. Cos $\theta = \frac{1}{3}$. Find csc θ.

Take $a = 3$ and $x = 1$ in Fig. 54. Find y and then write down the function from its definition.

3. Tan $\theta = 2$. Find sin θ.

Take $x = 1$ and $y = 2$ in Fig. 54, calculate a, and then write down the function from its definition.

4. Sec $\theta = 10$. Find csc θ.

Take $a = 10$ and $x = 1$ and compute y.

5. Find the values of all functions of θ if $\cot \theta = 1.5$.

6. Find the functions of θ if $\cos \theta = 0.1$.

7. Find the values of each of the remaining circular functions in each of the following cases:

(*a*) $\sin \theta = \frac{5}{13}$. (*d*) $\tan \theta = \frac{3}{4}$. (*g*) $\tan \theta = m$.

(*b*) $\cos \theta = \frac{4}{5}$. (*e*) $\sec \theta = 2$. (*h*) $\sin \theta = \dfrac{c}{\sqrt{c^2 + d^2}}$.

(*c*) $\sec \theta = 1.25$. (*f*) $\tan \theta = \frac{1}{3}$.

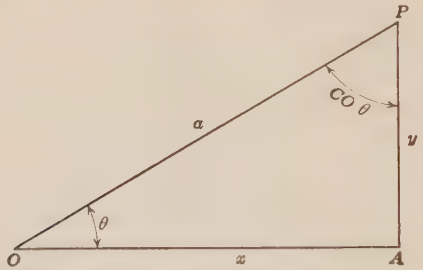


FIG. 54. Triangle of reference for θ and for complement of θ.

Show that the following equalities are correct:

8. $\tan \theta \cos \theta = \sin \theta$.

9. $\sin \theta \cot \theta \sec \theta = 1$.

10. $(\sin \theta + \cos \theta)^2 = 2 \sin \theta \cos \theta + 1$.

11. $\tan \theta + \cot \theta = \sec \theta \csc \theta$.

12. Express each trigonometric function in terms of each of the others; *i.e.*, fill in all blank spaces in the following table:

	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
$\sin \theta$	$\sin \theta$					$1/\csc \theta$
$\cos \theta$		$\cos \theta$			$1/\sec \theta$	
$\tan \theta$			$\tan \theta$	$1/\cot \theta$		
$\cot \theta$			$1/\tan \theta$	$\cot \theta$		
$\sec \theta$		$1/\cos \theta$			$\sec \theta$	
$\csc \theta$	$1/\sin \theta$					$\csc \theta$

The following exercises refer to angles $<360°$ of any quadrant:

13. If $\sin\theta = -\frac{3}{4}$ and $\tan\theta$ is positive, find the remaining five functions.

Hint: Since $\sin\theta$ is negative and $\tan\theta$ is positive, the angle θ is in the third quadrant (see Fig. 55).

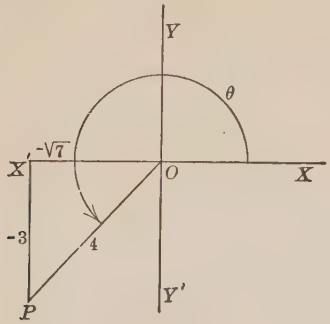


FIG. 55.—Diagram for exercise 13.

14. If $\cos\theta = 1\frac{2}{13}$ and $\sin\theta$ is negative, find the remaining five functions of θ.

15. If $\tan\theta = -\sqrt{3}$ and $\cos\theta$ is negative, find the remaining functions of θ.

16. If $\cos\theta = -\frac{1}{3}$ and $\sin\theta$ is positive, find the remaining functions.

17. If $\tan\theta = \frac{5}{12}$ and $\sec\theta$ is negative, find the remaining functions of θ.

18. If $\sin\theta = \frac{3}{5}$ and $\tan\theta$ is negative, find the remaining functions of θ.

57. Functions of Complementary Angles.—Angles are said to be **complementary** if their sum is 90°. Angles are said to be **supplementary** if their sum is 180°.

Let θ be an angle in the first quadrant, and draw the angle $(90° - \theta)$ of terminal side OP_1, as shown in Fig. 56. Let P and P_1 lie on a circle of radius a. Let the coordinates of the point P be (h, k), then P_1 is the point (k, h). Hence $P_1D_1/OP_1 = h/a = \sin(90° - \theta)$. But from the triangle PDO, $h/a = \cos\theta$. Hence

$$\sin(90° - \theta) = \cos\theta.$$

Likewise,

$$\tan (90° - \theta) = \cot \theta,$$
$$\sec (90° - \theta) = \csc \theta.$$

These relations explain the meaning of the words *cosine, cotangent, cosecant,* which are merely abbreviations for *complement's sine, complement's tangent, etc.* Collectively, cosine, cotangent, and cosecant are called the **cofunctions.** Likewise, from Fig. 56,

$$\cos (90° - \theta) = \sin \theta,$$
$$\cot (90° - \theta) = \tan \theta,$$
$$\csc (90° - \theta) = \sec \theta.$$

Later it will be shown that the above relations hold for all values of θ, positive or negative.

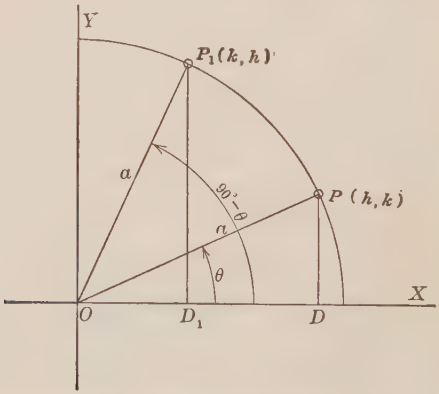


FIG. 56.—Triangles of reference for complementary angles.

58. Graph of the Sine and Cosine.—In rectangular coordinates we can think of the ordinate y of a point as depending for its value upon the abscissa or x of that point by means of the equation $y = \sin x$, provided we think of each value of the abscissa laid off on the X-axis as standing for some amount of angular magnitude. In order to produce the graph of $y = \sin x$, it is best to lay off the angular measure x on the X-axis in such a manner that it may conveniently be thought of in either radian or degree measure. If we suppose that a scale of inches and tenths is in the

hands of the student, and that a graph is required upon an ordinary sheet of unruled paper of letter size (8½ by 11 in.), then it will be convenient to let ⅕ in. of the horizontal scale of the X-axis correspond to 10° or to $\pi/18$ radians of angular measure. To accomplish this, the length of 1 radian must be 1.15 in. (*i.e.*, $18/5\pi$ in.), which length must be used for the radius of the circle on which the arcs of the angles are laid off (see Exercise 14, Sec. **47**). Hence, to graph $y = \sin x$, draw at the left of a sheet of (unruled) drawing paper a circle of radius 1.15 in., as the circle OP_5B (Fig. 57). Take O as the origin and prolong the radius BO for the positive

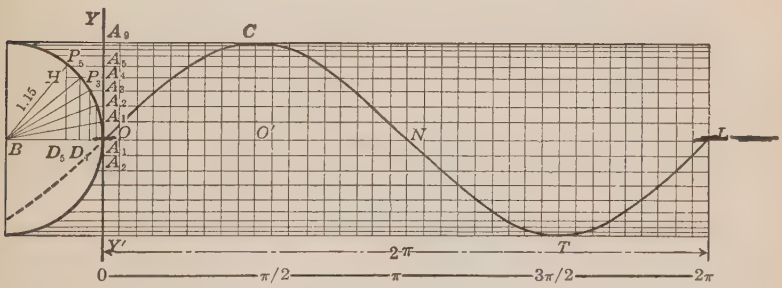


FIG. 57.—Construction of the sinusoid.

portion OX of the X-axis. Divide OL into ⅕-in. intervals, each corresponding to 10° of angle; eighteen of these correspond to the length π, if the radius BO (1.15 in.) be the unit of measure. Next divide the Y-axis proportionately to $\sin x$ in the following manner: With a pair of bow dividers, or by means of a protractor, divide the semi-circle into 18 equal divisions as shown in the figure, thus making the length of each small arc exactly ⅕ in. The perpendiculars, or ordinates, dropped upon OX from each point of division, divided by the radius, are the sines of the corresponding angles. Draw lines parallel to OX through each point of division of this circle.[1] These cut the Y-axis at points A_1, A_2 . . . Then if the radius of the circle be called unity, the distances OA_1, OA_2, OA_3, . . . are, respectively, the sines of the angles OBP_1, OBP_2, OBP_3, . . . These are the successive ordinates corresponding to the abscissas

[1] For lack of room only a few of the successive points P_1, P_2, P_3, . . . of division of the quadrant of OP_3P_5 are actually lettered in Fig. 57.

already laid off on OL. The curve is then constructed as follows: First draw vertical lines through the points of division of OX; these, with the horizontal lines already drawn, divide the plane into a large number of rectangles. Starting at O and sketching the diagonals (curved to fit the alignment of the points) of successive "cornering" rectangles, the curve $OCNTL$ is approximated, which is the graph of $y = \sin x$. This curve is called the **sinusoid** or **sine curve.** The curve is of very great importance, for it is found to be the *type form* of the fundamental waves of science, such as sound waves, vibrations of wires, rods, plates and bridge members, tidal waves in the ocean, and ripples on a water surface. The ordinary progressive waves of the sea are, however, not of this shape. Using terms borrowed from the language of waves, we may call C the **crest,** N the **node,** and T the **trough** of the sinusoid.

It is obvious that, as x increases beyond $2\pi^r$, the curve is repeated, and that the pattern $OCNTL$ is repeated again and again both to the left and to the right of the diagram as drawn. Thus it is seen that the sine is a **periodic function** of period $2\pi^r$, or 360°.

The small rectangles lying along the X-axis are *nearly squares.* They would be *exactly* equilateral if the straight line OA_1 was equal to the arc OP_1. This equality is approached as near as we please as the number of corresponding divisions of the circle and of OX is indefinitely increased. In this way we arrive at the notion of the **slope of a curve** in mathematics. In this case we say that the **slope of the sinusoid** at O is $+1$ and at N is -1, and at L is $+1$. We say that the curve cuts the axis at an angle of 45° at O and at an angle of 315° (or $-45°$ if we prefer) at N. The slope at C and at T is zero.

The student should note that, in Fig. 57, there are in reality two scales indicated on the X-axis, to be used as the angle is thought of as measured in radians or in degrees respectively. The division points through which the vertical lines are drawn belong to the degree scale. The numbers $\pi/2$, π, $3\pi/2$, 2π belong to the radian scale.

The curve $y = a \sin x$ is made from $y = \sin x$ by multiplying all of the ordinates of the latter by a. The number a is called the **amplitude** of the sinusoid.

59. The Cosine Curve.—In Fig. 58 let the angles COP_1, COP_2, COP_3, etc. be laid off from the position of the Y-axis OY as initial side. Then if the radius of the circle be called unity, the distances OD_1, OD_2, OD_3, . . . are, respectively, the cosines of the angles COP_1, COP_2, COP_3, . . . If the distances laid off on the X-axis represent the measures of the successive angles COP_1, COP_2, . . . then the curve shown in the figure has the equation $y =$

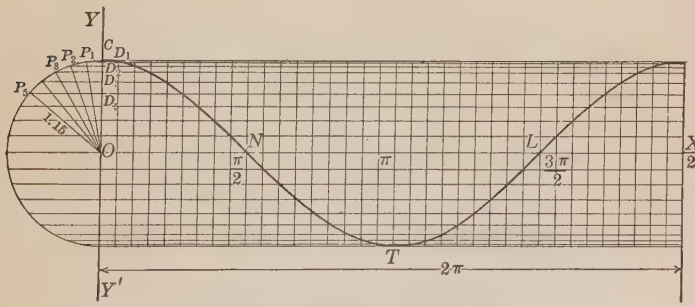


FIG. 58.—Construction of the cosine curve.

cos x. The construction shows that the curve is exactly the same as the sine curve of Fig. 57, except that the origin for the cosine curve is under the *crest* while in the sine curve the origin is at a *node*. If the origin be taken at O' in Fig. 57, the curve may be called the cosine curve.

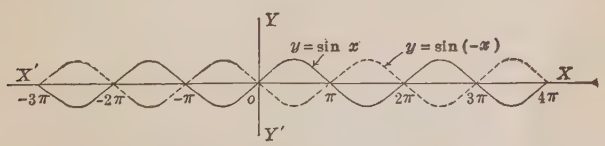


FIG. 59.—The relation between $y = \sin x$ and $y = \sin(-x)$.

60. The Sine of a Negative Angle.—In Fig. 59 the full-drawn curve represents the graph for $y = \sin x$. The graph for

$$y = \sin(-x) \tag{1}$$

may be obtained by rotating the graph for $y = \sin x$, 180° about the Y-axis, by Theorem I on Loci. This gives the dotted curve of Fig. 59. But, from the properties of the sinusoid, the dotted curve

is the reflection in the X-axis of the curve drawn in full; hence the equation of the dotted curve may also be written

$$-y = \sin x. \tag{2}$$

Hence, from Eqs. (1) and (2)

$$\sin(-x) = -\sin x. \tag{3}$$

61. Complementary Angles.—Figure 60 shows the curves for $y = \cos x$ and for $y = \sin x$. By the properties of these curves it is obvious that the cosine curve may be regarded as the sine curve translated $\pi/2$ units to the left. That is, the cosine curve

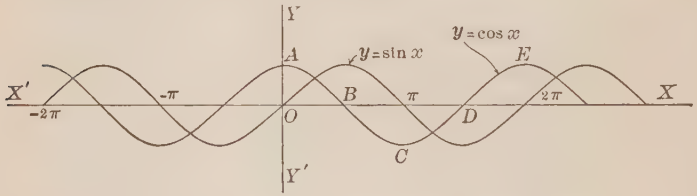


FIG. 60.—Comparison of the sine and cosine curves.

$$y = \cos x \tag{1}$$

has also the equation

$$y = \sin\left(x + \frac{\pi}{2}\right). \tag{2}$$

Since this curve (the cosine curve) is symmetrical about the Y-axis, its equation remains unchanged if we change x to $(-x)$, by Theorem I on Loci. Hence the cosine curve has also the equation

$$y = \sin\left(-x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - x\right). \tag{3}$$

Comparing (1) and (3) we see that we have proved for all values of x that

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x. \tag{4}$$

By comparing Eqs. (1) and (2) we see that $\sin\left(\frac{\pi}{2} + x\right) = \cos x$; this fact is, however, much less useful than that represented by Eq. (4).

Exercises

From the curves for $y = \sin x$ and $y = \sin(-x)$, Fig. 59 shows that:

1. $\sin(x - \pi) = \sin(-x)$ and hence $\sin(\pi - x) = \sin x$.

From the curves for $y = \cos x$ and $y = \sin x$, Fig. 60 shows that:

2. $\sin x = \cos\left(x - \frac{\pi}{2}\right)$.

3. $\cos x = \sin(x - \frac{3}{2}\pi)$.

4. $\cos(x + \frac{3}{2}\pi) = \sin x$.

5. $\cos(-x) = \cos x$.

62. Trigonometric Functions of Negative Angles.—We have already shown, in Eq. (3), Sec. **60,** that

$$\sin(-x) = -\sin x. \tag{1}$$

Also from Fig. 60, since the cosine curve is symmetrical about the Y-axis,

$$\cos(-x) = \cos x. \tag{2}$$

Dividing the members of Eq. (1) by the members of Eq. (2) we find

$$\tan(-x) = -\tan x. \tag{3}$$

63. Odd and Even Functions.—A function that changes sign but retains the same numerical value when the sign of the argument is changed is called an **odd function.** Thus, $\sin x$ is an odd function of x, since $\sin(-x) = -\sin x$. Likewise, x^3 is an odd function of x, as are all odd powers of x. The graph of an odd function of x is symmetrical with respect to the origin O; that is, if P is a point on the curve, then if the line OP be produced backward through O a distance equal to OP to a point P', then P' also lies on the curve. The parts of $y = x^3$ in the first and third quadrants are good illustrations of this property.

A function of x that remains unaltered, both in sign and in numerical value, when the argument is changed in sign, is called an **even function** of x. Examples are $\cos x$, x^2, $x^2 - 3x^4$, . . . The graph of an even function is symmetrical with respect to the Y-axis.

Most functions are neither odd nor even, but mixed, like $x^2 + \sin x$, $x^2 - x^3$, and $x + \cos x$.

Exercises

1. Is $\sin^2 x$ an odd or an even function of x? Is $\tan^3 x$ an odd or an even function of x?

2. Is the function $\sin x + 2 \tan x$ an odd or an even function? Is $\sin x + \cos x$ an odd or an even function of x?

64. Polar Coordinates.—In Sec. **10,** where coordinate systems were first discussed, it was shown that any system whereby two numbers locate a point in a plane may be used as a basis for the graphic representation of functional relationships; and the rectangular and polar systems of coordinates were given as examples.

In Fig. 61, the position of the point P may be assigned either by giving the x and y of the rectangular coordinate system, or by giving the **vectorial angle** θ and the distance OP measured along the terminal side of θ. Unlike the distance a used in the preceding work, it is found convenient to give the line OP a **sense** or **direction** as well as length; such a line is called a **vector.** In the present case, OP is known as the **radius vector** of the point P, and it is usually symbolized by the letter ρ. The vectorial angle θ and the radius vector ρ are together called the **polar coordinates** of the point P, and the system used in locating the point is known as the **system of polar coordinates.** In Fig. 61 the point P' is located by turning from the fundamental direction OX, called the **polar axis,** through an angle θ and then stepping backward the distance ρ to the point P'; this is, then, the point $(-\rho, \theta)$. P' has also the coordinates (ρ, θ_2), in which $\theta_2 = \theta + 180°$; likewise, P_1 is $(+\rho', \theta_1)$ and P'_1 is $(-\rho', \theta_1)$. Thus each point may be located in the polar system of coordinates in two ways, *i.e.*, with either a positive or a negative radius vector. If negative values of θ be used, there are *four* ways of locating a point without using values of $\theta > 360°$. In giving a point in polar coordinates, it is usual to name the radius vector first and then the vectorial angle; thus (5, 40°) means the point of radius vector 5 and vectorial angle 40°.

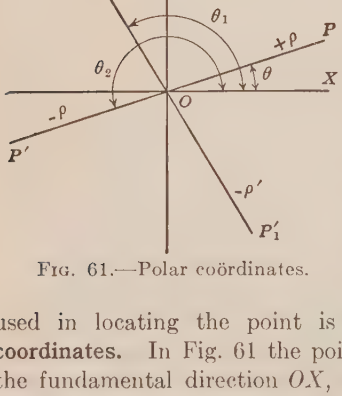


FIG. 61.—Polar coördinates.

In rectangular coordinates it is usual to use x to represent the independent variable, and y the dependent, so that a general functional relationship is written $y = f(x)$. In polar coordinates it is usual to use θ to represent the independent variable, and ρ the dependent, so that a general functional relationship will usually be written in the form $\rho = f(\theta)$.

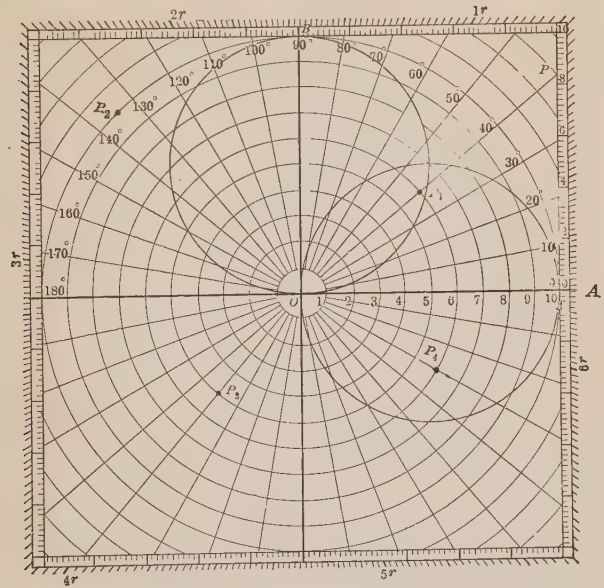


FIG. 62.—Polar coordinate paper.

65. Polar Coordinate Paper.—Polar coordinate paper (form *M*3) is prepared for the construction of loci in the polar system. A reduced copy of a sheet of such paper is shown in Fig. 62. This plate is graduated in degrees, but a scale of radian measure is given in the margin. The radii proceeding from the pole *O* meet the circles at right angles, just as the two systems of straight lines meet each other at right angles in rectangular coordinate paper. For this reason, both the rectangular and the polar systems are called **orthogonal systems of coordinates.**

We have learned that the fundamental notion of a function implies a table of corresponding values for two variables, one called the argument and the other the function. The notion of a graph implies any sort of a scheme for a pictorial representation of this table of values. There are three common methods in use: the double scale, the rectangular coordinate paper, and the polar paper. The polar paper is very convenient in case the argument is an angle measured in degrees or in radians. Since in a table of values for a functional relation we need to consider both positive and negative values for both the argument and the function, it is necessary to use on the polar paper the convention already explained. The argument, which is the angle, is measured counter-clockwise if positive and clockwise if negative from the line numbered 0° (Fig. 62). The function is measured outward from the center along the terminal side of the angle for positive functional values and outward from the center along the terminal side of the angle *produced backward through the center* for negative functional values. In this scheme it appears that four different pairs of values are represented by the same point. This is made clear by the points plotted in the figure. The points P_1, P_2, P_3, P_4 are as follows:

P_1: (6, 40°); (6, −320°); (−6, 220°); (−6, −140°).
P_2: (10, 135°); (10, −225°); (−10, 315°); (−10, −45°).
P_3: (5, 230°); (5, −130°); (−5, 50°); (−5, −310°).
P_4: (6, 330°); (6, −30°); (−6, 150°); (−6, −210°).

The angular scale cannot be changed, but the functional scale can be changed at pleasure.

In case the vectorial angle is given in radians, the point may be located on polar paper by means of a straight edge and the marginal scale on form *M*3.

The point *O* (Fig. 62) is called the **pole** and the line *OA* the **polar axis.**

Exercises

1. Plot upon polar coordinate paper the following: (*a*) (0.1, 30°); (*b*) (0.2, 40°); (*c*) (0.6, 120°); (*d*) (0.8, −30°); (*e*) (1.2, 300°); (*f*) (0.7, −47°). Let 10 cm. = 1 unit for ρ.

2. Plot upon polar coordinate paper the following: (*a*) (1.3, 45°); (*b*) (11.1, 137°); (*c*) (9.2, −47°); (*d*) (8.5, −216°). Let 1 cm. = 1 unit for ρ.

3. Plot upon polar coordinate paper the following: (*a*) (10, 0^r); (*b*) (9, 1^r); (*c*) (8.2, 1.6^r); (*d*) (12, 3.2^r). Let 1 cm. = 1 unit for ρ.

4. Explain why the locus for $\rho = 3$ is a circle with center at the pole and radius equal to 3 units.

5. Draw the loci for $\rho = 5$ and $\rho = 7$.

6. Explain why the locus $\theta = \frac{1}{4}\pi$ is a straight line passing through the pole and making an angle of 45° with the polar axis. Explain why this locus is indefinite in extent and does not terminate at the pole.

7. Draw loci for: $\theta = \frac{3}{4}\pi$, and $\theta = -\frac{1}{4}\pi$.

8. Plot the locus for $\rho = \theta$, if θ is measured in radians. Use 2 cm. as the unit for ρ.

66. Graphs of $\rho = a \cos \theta$ and $\rho = a \sin \theta$.—The two equations $y = \cos x$ and $\rho = \cos \theta$ are alternative expressions for the same functional relationship, namely, that which exists between an angle and its cosine. The graphic representation of this relationship, however, differs in the case of the two equations, since the notation used in the first equation indicates the use of rectangular coordinates, while the notation in the second equation indicates the use of polar coordinates. To graph the equation $\rho = \cos \theta$ it is only necessary to locate a sufficient number of points by means of pairs of corresponding values of ρ and θ, and to draw a smooth curve through the points thus obtained. These pairs of values could be taken from a table such as that given, for angles in the first quadrant, by the first and fourth columns of Table IV, Sec. **50.** An alternative method is to use the Cartesian graph of the equation $y = \cos x$ already prepared by the student in connection with his study of Sec. **59.** Making use of this latter suggestion, one would obtain the polar graph of the equation $\rho = \cos \theta$ as follows: Through the middle point O of an unruled sheet of drawing paper, draw a line, to be used as the polar axis, parallel to the longer side of the paper. Through the origin O draw light radiating lines making angles with the polar axis of 10°, 20°, 30°, 40°, etc. With a pair of dividers lay off on these lines the cosines of the angles 10°, 20°, 30°, 40°, etc., taken directly from the Cartesian graph of $y = \cos x$. There will be thus obtained a series of points, one corresponding to every 10° between 0° and 360° through which the curve may be drawn. At the same time these points are being laid off, the student can determine a second set of points by laying off on the radiating lines *twice* the values of the cosines as taken from the Cartesian graph. These are evidently points on the polar graph of the equation $\rho = 2 \cos \theta$.

In just the same way the equations $\rho = \sin\theta$, $\rho = 2\sin\theta$, or, in general, $\rho = a\sin\theta$ may be graphed, making use of the Cartesian graph of $y = \sin x$.

The method of constructing these polar graphs should make it clear that they contain the same information as do the corresponding Cartesian graphs, although they differ greatly from them in appearance. If the work has been carefully done, the student finds that the polar curves $\rho = a\cos\theta$ and $\rho = a\sin\theta$ are circles of diameter a passing through the origin. The former lies entirely to the right of the line $\theta = 90°$, while the latter lies entirely above the polar axis. The graphic construction does not prove this statement, but indicates the probability of its truth. The student can however, easily prove that any point P located by a pair of

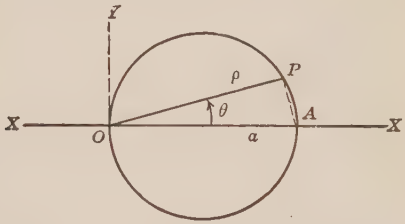


FIG. 63.—Graph of $\rho = a\cos\theta$.

values of ρ and θ which satisfy the equation $\rho = a\cos\theta$ lies on the circle of diameter a shown in Fig. 63. He need use only one of the rules of Sec. **52,** and the proposition of plane geometry that a right-angled triangle is inscribable in a semi-circle. A still more simple proof will be indicated later, in Exercises 9 and 10 of Sec. **72.** Both the above construction and the more general proof should make it clear that, as θ varies from 0° to 360°, the circle is twice drawn.

The cosine of a variable, if graphed in rectangular coordinates, is a sinusoid; but, if graphed in polar coordinates, it is a circle, twice drawn. There is in this case a very great difference in the ease with which these curves can be constructed; the sinusoid requires an elaborate method, while the circle can be drawn at once with compasses. This is one reason why the periodic or

sinusoidal relation is preferably represented, in the natural sciences, by polar coordinates.

Exercises

1. Show that if $-a$ be negative, $\rho = -a \cos\theta$ is a circle, diameter a, with center to left of the pole $\frac{1}{2}a$ units.

2. Show that if $-a$ be negative $\rho = -a \sin\theta$ is a circle, diameter a, with center below the pole $\frac{1}{2}a$ units.

67. Graphic Table of Sines and Cosines.—The polar graphs of $\rho = a \sin\theta$ and $\rho = a \cos\theta$ furnish the best means of constructing graphic tables of sines and cosines. The two circles passing through O shown on the polar coordinate paper (form $M3$, Fig. 62) are drawn for this purpose. A supply of this coordinate paper should be in the hands of the student. If the diameter of the sine and cosine circles be called 1, then the radius vector of any point on the lower circle is the cosine of the vectorial angle, and the radius vector of the corresponding point on the upper circle is the sine of the vectorial angle. Thus, from the diagram of form $M3$, we read $\cos 45° = 0.707$; $\cos 60° = 0.500$; $\cos 30° = 0.866$. These results are correct to the third place.

Exercises

From coordinate paper (form $M3$) find the values of the following: (*a*) $\cos 36°$; (*b*) $\cos 62°$; (*c*) $\cos 126°$; (*d*) $\sin 81°$; (*e*) $\sin 25°$; (*f*) $\sin 226°$.

68. Graphic Table of Tangents and Secants.—Referring to Fig. 62, it is obvious that the numerical values of the tangents of angles can be read off by use of the uniform scale bordering the polar paper (form $M3$). The scale referred to lies just inside of the scale of radian measure, and is numbered 0, 2, 4, . . . Thus to get the numerical value of tan 40° it is merely necessary to call unity the side OA of the triangle of reference OAP, and then read the side $AP = 0.84$; hence $\tan 40° = 0.84$. To the same scale (*i.e.*, $OA = 1$) the distance $OP = 1.31$, but this is the secant of the angle AOP, whence $\sec 40° = 1.31$. By use of the circles we find $\sin 40° = 0.64$ and $\cos 40° = 0.76$.

In case we are given an angle greater than 45° (but less than 135°), use the horizontal scale through B. Starting from B as zero, the distance measured on the horizontal scale is the *cotangent* of the given angle. The tangent is found by taking the reciprocal of the cotangent.

Exercises

Find the unknown sides and angles in the following right triangles. The numerical values of the trigonometric functions may be taken from the polar paper. The vertices of the triangles are supposed to be lettered A, B, C, with C at the vertex of the right angle. The small letters a, b, c represent the sides opposite the angles of the same name. See also table of Natural Trigonometric Functions at end of the book.

By **angle of elevation** of an object is meant the angle between a horizontal line and a line to the object, both drawn from the point of observation, when the object lies *above* the horizontal line. The similar angle when the object lies *below* the observer is called the **angle of depression** of the object.

The solution of each of the following problems must be checked. The easiest check is to draw the triangles accurately to scale on form $M1$, measuring the unknown sides and angles.

1. When the altitude of the sun is 40°, the length of the shadow cast by a flag pole on a horizontal plane is 90 ft. Find the height of the pole.

Outline of Solution: Call height of pole a, and length of shadow b. Then $A = 40°$ and $B = 50°$. Hence,

$$a = b \tan 40°.$$

Determining the numerical value of the tangent from the polar paper, we find

$$a = 90 \times 0.84 = 75.6 \text{ ft.},$$

which result, if checked, is the height of the pole. To check, either draw a figure to scale, or compute the hypotenuse c, thus:

$$c = 90 \sec 40°.$$

From the polar paper find sec 40°. Then

$$c = 90 \times 1.31 = 117.9.$$

Since $a^2 + b^2 = c^2$, we have $c^2 - b^2 = a^2$, or $(c - b)(c + b) = a^2$. Hence, if the result found be correct,

$$(117.9 - 90)(117.9 + 90) = (75.6)^2$$
$$5800 = 5715.$$

These results show that the work is correct to about three figures, for the sides of the triangle are proportional to the *square roots* of the numbers last given.

2. At a point 200 ft. from, and on a level with, the base of a tower the angle of elevation of the top of the tower is observed to be 60°. What is the height of the tower?

3. A ladder 40 ft. long stands against a building with the foot of the ladder 15 ft. from the base of the wall. How high does the ladder reach on the wall?

4. From the top of a vertical cliff the angle of depression of a point on the shore 150 ft. from the base of the cliff is observed to be 30°. Find the height of the cliff.

5. In walking half a mile up a hill, a man rises 300 ft. Find the angle at which the hill slopes.

If the hill does not slope uniformly the result is the *average slope* of the hill.

6. A line 3.5 in. long makes an angle of 35° with OX. Find the lengths of its projections upon both OX and OY.

7. A vertical cliff is 425 ft. high. From the top of the cliff the angle of depression of a boat at sea is 16°. How far is the boat from the foot of the cliff?

8. The projection of a line on OX is 7.5 in., and its projection on OY is 1.25 in. Find the length of the line, and the angle it makes with OX.

9. A battery is placed on a cliff 510 ft. high. The angle of depression of a floating target at sea is 9°. Find the range, or the horizontal distance of the target from the battery.

10. From a point A the angle of elevation of the top of a monument is 25°. From the point B, 110 ft. farther away from the base of the monument and at the same elevation as A, the angle of elevation is 15°. Find the height of the monument above the line AB.

11. Find the length of a side of a regular pentagon inscribed in a circle whose radius is 12 ft.

12. Proceeding south on a north-and-south road, the direction of a church tower, as seen from a milestone, is 41° west of south. From the next milestone the tower is seen at an angle of 65° west of south. Find the shortest distance of the tower from the road.

13. Observations of the height of a mountain were taken at A and B on the same horizontal line, and in the same vertical plane with the top of the mountain. The elevation of the top at A is 52° and at B is 36°. The distance AB is 3500 ft. Find the height of the mountain.

14. The diagonals of a rhombus are 16 and 20 ft. Find the lengths of the sides and the angles of the rhombus.

15. The equation of a line is $y = \frac{3}{4}x + 10$. Compute the shortest distance of this line from the origin.

16. Find the perimeter and area of $ABCD$ (Fig. 64).

17. Find BC and the total area of $ABCD$ (Fig. 65).

69. The Law of the Circular Functions.—It will be emphasized in this book that the fundamental laws of exact science are three in number, namely: (1) the power function expressed by $y = ax^n$, where n may be either positive or negative; (2) the harmonic or periodic law $y = a \sin nx$, which is fundamental to all periodically

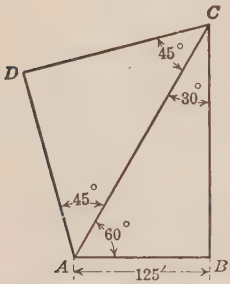


Fig. 64.—Diagram for Exercise 16.

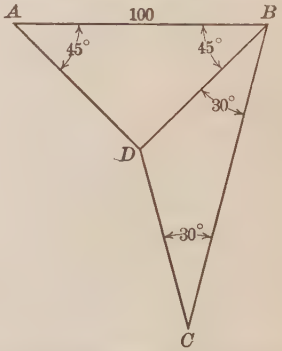


Fig. 65.—Diagram for Exercise 17.

occurring phenomena; and (3) a law to be discussed in a subsequent chapter. While other important laws and functions arise in the exact sciences, they are secondary to those expressed by the three fundamental relations.

We have stated the law of the power function in the following words (see Sec. **35**):

In any power function, if x change by a fixed multiple, y changes by a fixed multiple also. In other words, if x change by a constant factor, y will change by a constant factor also.

Confining our attention to the fundamental functions, sine and cosine, in terms of which the other circular functions can be expressed, we may state their law as follows:[1]

The circular functions, sin θ and cos θ, change periodically in value proportionally to the periodic change in the ordinate and abscissa, respectively, of a point moving uniformly on the circle $x^2 + y^2 = a^2$.

The use of the periodic law in the natural sciences is, of course, very different from that of the power function. The student will find that circular functions similar to $y = a \sin nx$ will be required in order to express properly all phenomena which are recurrent or periodic in character, such as the motion of vibrating bodies, all forms of wave motion, such as sound waves, light waves, electric waves, alternating currents and waves on water surfaces, etc. Almost every part of a machine, no matter how complicated its motions, repeats its original motions at stated intervals and these recurrent positions are expressible in terms of the circular functions and not otherwise. The student will obtain a very limited and unprofitable idea of the use of the circular functions if he deems that their principal use is in numerical work in solving triangles, etc. The importance of the circular functions lies in the power they possess of expressing natural laws of a periodic character.

70. Rotation of Any Locus.—In Sec. **36** we have shown that any locus $y = f(x)$ is translated a distance a in the x direction by substituting $x - a$ for x in the equation of the locus. Likewise, the substitution of $y - b$ for y was found to translate the locus the distance b in the y direction. A discussion of the rotation of a locus was not considered at that place, because a displacement of this type is best brought about when the equations are expressed in polar coordinates.

If a table of values be prepared for each of the loci,

$$\rho = \cos \theta \tag{1}$$

$$\rho = \cos (\theta - 30°) \tag{2}$$

as follows:

[1] Chapter X is devoted to a discussion of these fundamental periodic laws.

Equation 1		Equation 2	
θ	ρ	θ	ρ
$-30°$	0.866	$0°$	0.866
$-20°$	0.940	$10°$	0.940
$-10°$	0.985	$20°$	0.985
$0°$	1.000	$30°$	1.000
$10°$	0.985	$40°$	0.985
$20°$	0.940	$50°$	0.940
$30°$	0.866	$60°$	0.866
$40°$	0.766	$70°$	0.766
$50°$	0.643	$80°$	0.643
$60°$	0.500	$90°$	0.500

and then if the graph of each be drawn (Fig. 66) it will be seen that the curves differ only in location and not at all in shape or size.

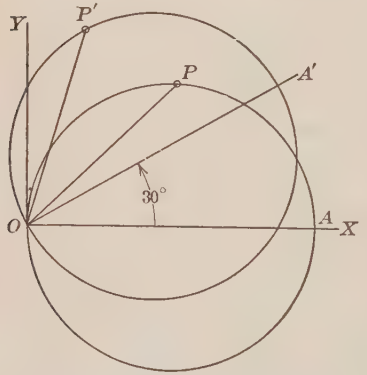


FIG. 66. — Rotation of the circle OAP [$\rho = a \cos \theta$] to the position $OA'P'$ [$\rho = a \cos (\theta - 30°)$].

If a value be given to θ_1 in the second equation which is 30° greater than a value given to θ in the first equation, the two values of ρ from Eqs. (1) and (2) are equal. Thus, if AOP be the value given to θ in Eq. (1) and if $AOP' = AOP + 30°$ be the value given to θ_1 in Eq. (2), then OP will equal OP'. Thus the point P' may be looked upon as having been obtained from the point P by a positive rotation about O of 30°. Thus, the graph for $\rho = \cos (\theta - 30°)$ may be obtained from the graph for $\rho = \cos \theta$ by rotating it about the pole O through an angle of 30°.

The same reasoning will apply if $\theta - \alpha$ be substituted for θ; in this case the locus of the first curve is rotated about the pole through an angle α—in the positive sense if α be positive, in a negative sense if α be negative.

By the same reasoning as used above, we see that if, in the polar equation of *any* curve, θ be replaced by $(\theta - \alpha)$, the graph of the new equation is the graph of the original equation rotated about the pole through an angle α, but otherwise unchanged.

THEOREMS ON LOCI

IX. *If* $(\theta - \alpha)$ *be substituted for* θ *throughout the polar equation of any locus, the curve is rotated through the angle* α.

Note that the rotation is positive when α is positive and negative when α is negative.

Exercises

1. Upon a sheet of polar coordinate paper draw: $\rho = \cos\theta$; $\rho = \cos(\theta - 60°)$; $\rho = \cos(\theta + 60°)$.

2. Upon a sheet of polar coordinate paper draw: $\rho = \sin\theta$; $\rho = \sin(\theta - 30°)$; $\rho = \sin(\theta + 30°)$.

3. Upon a sheet of polar coordinate paper draw: $\rho = \cos\theta$; $\rho = \cos\left(\theta - \frac{\pi}{2}\right)$; $\rho = \cos\left(\theta + \frac{\pi}{2}\right)$; $\rho = \cos(\theta - \pi)$.

4. Prove the relationship $\cos(\theta + 90°) = -\cos(\theta - 90°)$ by comparing the circles $\rho = \cos(\theta + 90°)$ and $\rho = -\cos(\theta - 90°)$.

71. The Graph of the Equation $a = \rho\cos(\theta - \alpha)$.—From the equation $\rho = 5/\cos(\theta - 30°)$ or $5 = \rho\cos(\theta - 30°)$ there can be computed pairs of corresponding values of ρ and θ, which can be used to locate points on the graph of this equation. The following table gives the values of ρ for values of θ between $-30°$ and $100°$, as obtained from this equation.

CORRESPONDING VALUES OF ρ AND θ AS COMPUTED FROM THE EQUATION $5 = \rho\cos(\theta - 30°)$

θ	ρ	θ	ρ
$-30°$	10.00	$40°$	5.08
$-20°$	7.78	$50°$	5.32
$-10°$	6.53	$60°$	5.77
$0°$	5.77	$70°$	6.53
$+10°$	5.32	$80°$	7.78
$20°$	5.08	$90°$	10.00
$30°$	5.00	$100°$	14.60

The student should use this table (choosing 1 cm. as a unit for ρ) to construct a polar graph of the equation $5 = \rho \cos (\theta - 30°)$. He will find that the points lie on a straight line. What is the distance from the pole to this line? What angle does the line make with the polar axis? What angle is made with the polar axis by the perpendicular dropped on the line from the pole?

The student should prove, using Fig. 67, that the information obtained from the above special example is entirely general. That

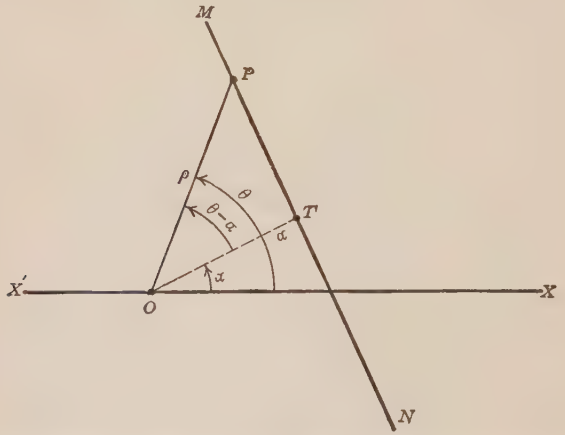


FIG. 67.—Diagram for use in proving that the graph of $a = \rho \cos (\theta - \alpha)$ is a straight line.

is, he should prove that the polar graph of the equation $a = \rho \cos (\theta - \alpha)$ is a straight line MN, and that the normal OT dropped from the pole on the line is of length a and makes an angle α with the polar axis. A simpler proof, using another method, will be later indicated, in Exercise 11, Sec. **72.**

The equation of any straight line drawn upon the polar plane can be written in the form $a = \rho \cos (\theta - \alpha)$ by assigning proper values to a and α. For example, a line parallel to the polar axis but 3 units above it has, for its equation, $3 = \rho \cos (\theta - 90°)$. The equation $a = \rho \cos (\theta - \alpha)$ is called the **normal equation of the straight line.**

Exercises

1. Write the polar equation of a line tangent to the circle $\rho = 5 \cos (\theta - 30°)$ at the end of the diameter passing through the pole.

2. A line is 3 units distant from the pole and makes an angle of 45° with the polar axis. Write its polar equation.

3. Describe curves $\rho = 10 \cos \left(\theta - \frac{\pi}{4}\right)$ and $10 = \rho \cos \left(\theta - \frac{\pi}{4}\right)$.

Draw the following circles:

4. $\rho = 3 \cos (\theta - 30°)$.

5. $\rho = 3 \cos (\theta + 120°)$.

6. $\rho = 2 \sin (\theta - 45°)$.

7. $\rho = 2 \sin (\theta + 135°)$.

8. $\rho = 4 \cos \left(\theta + \frac{\pi}{3}\right)$.

9. $\rho = 5 \sin \left(\frac{\pi}{2} - \theta\right)$.

10. Show that $\rho = a \sin \theta$ is the locus $\rho = a \cos \theta$ rotated 90° counter-clockwise.

11. Why is the function $y = mx + b$ called a linear function? If polar coordinates were considered the most fundamentally important type of coordinates, what function would it be natural to call a linear function?

72. Relation between Rectangular and Polar Coordinates.— Think of the point P (Fig. 68) whose rectangular coordinates are (x, y). If the radius vector OP be called ρ and its direction angle be called θ, then the polar coordinates of P are (ρ, θ). Then x and y, for any position of P, are the projections of ρ through the angle θ, and the angle $(90° - \theta)$, respectively, or

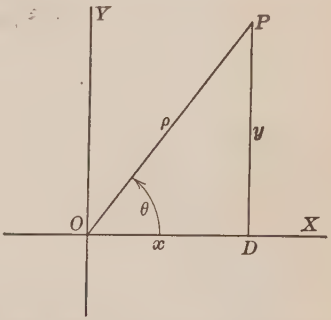


FIG. 68.—Rectangular and polar coördinates of a point P.

$$x = \rho \cos \theta, \tag{1}$$

$$y = \rho \sin \theta. \tag{2}$$

These are the equations of transformation which enable us to write the equation of a curve in polar coordinates when its equation in rectangular coordinates is known, or *vice versa*. Thus the straight line $x = 3$ has the equation

$$\rho \cos \theta = 3$$

in polar coordinates. The line $x + y = 3$ has the polar equation

$$\rho \cos \theta + \rho \sin \theta = 3.$$

The circle $x^2 + y^2 = a^2$ has the equation

$$\rho^2 \cos^2 \theta + \rho^2 \sin^2 \theta = a^2,$$

or

$$\rho^2 = a^2,$$

or

$$\rho = a.$$

To solve Eqs. (1) and (2) for θ, we write

$$\theta = \text{the angle whose cosine is } \frac{x}{\rho},$$

$$\theta = \text{the angle whose sine is } \frac{y}{\rho}.$$

The verbal expressions "the angle whose cosine is," etc. are abbreviated in mathematics by the notations "arc cos," read "arc-cosine," and "arc sin," read "arc-sine," as follows:

$$\theta = \textbf{arc cos} \left(\frac{x}{\rho}\right). \tag{3}$$

$$\theta = \textbf{arc sin} \left(\frac{y}{\rho}\right). \tag{4}$$

Dividing the members of Eq. (2) by the members of Eq. (1) we obtain $\tan \theta = y/x$, which, solved for θ, we write

$$\theta = \text{the angle whose tangent is } \frac{y}{x},$$

which may be abbreviated

$$\theta = \textbf{arc tan} \left(\frac{y}{x}\right) \tag{5}$$

and read "θ = the arc-tangent of y/x."

The value of ρ in terms of x and y is readily written

$$\rho = \sqrt{x^2 + y^2}. \tag{6}$$

Exercises

1. Write the polar equation of $x^2 + y^2 + 8x = 0$.
The result is $\rho^2 + 8\rho \cos \theta = 0$, or $\rho = -8 \cos \theta$.

2. Write the polar equations of: (*a*) $x^2 + y^2 - 4y = 0$; (*b*) $x^2 + y^2 - 6x - 4y = 0$; (*c*) $x^2 + y^2 - 6y = 4$.

3. Write the polar equations of: (*a*) $x + y = 1$; (*b*) $x + 2y = 1$; (*c*) $x + \sqrt{3}y = 2$.

4. Write the rectangular equations of: (*a*) $\rho \cos \theta + \rho \sin \theta = 4$; (*b*) $\rho \cos \theta - 3\rho \sin \theta = 6$.

5. Write the polar equation of $x^2 + 2y^2 - 4x = 0$.

6. Write the rectangular equation of $\rho = 2 \cos \theta + 3 \sin \theta$.

Hint: Multiply both members of the equation by ρ, replace ρ^2 by $(x^2 + y^2)$, $\rho \cos \theta$ by x, and $\rho \sin \theta$ by y.

7. Write the rectangular equation of $\rho = 3 \cos \theta - 2 \sin \theta$

8. Write the rectangular equation of $\rho = 5 \sin \theta - 3 \cos \theta$.

9. Write the rectangular equation of a circle with radius $a/2$ whose center is at the point $(a/2, 0)$. Change to polar coordinates. Compare with Sec. **66.**

10. Write the rectangular equation of a circle with radius $a/2$ whose center is at the point $(0, a/2)$. Change to polar coordinates. Compare with Sec. **66.**

11. Change the equation $a = \rho \cos \theta$ to rectangular coordinates. By a comparison of $a = \rho \cos (\theta - \alpha)$ with $a = \rho \cos \theta$ it is then possible to prove that the graph of $a = \rho \cos (\theta - \alpha)$ is a straight line.

73. Graph of $\rho = a \cos \theta + b \sin \theta$.—Before reading this section the student should review Exercises 6 and 7 of the preceding section. Let us find the Cartesian equation for the curve whose polar equation is

$$\rho = a \cos \theta + b \sin \theta, \tag{1}$$

where a and b are any constants, positive or negative. First multiply each member of Eq. (1) by ρ.

$$\rho^2 = a\rho \cos \theta + b\rho \sin \theta. \tag{2}$$

Since $\rho^2 = x^2 + y^2$, $\rho \cos \theta = x$, and $\rho \sin \theta = y$, Eq. (2) becomes

$$x^2 + y^2 = ax + by. \tag{3}$$

Transposing and completing squares,

$$\left[x - \frac{a}{2}\right]^2 + \left[y - \frac{b}{2}\right]^2 = \frac{a^2 + b^2}{4}. \tag{4}$$

This is the Cartesian equation of a circle with center at the point $(\frac{1}{2}a, \frac{1}{2}b)$ and of radius $\frac{1}{2}\sqrt{a^2 + b^2}$. The circle passes through the pole or origin since the coordinates (0, 0) satisfy Eq. (3), and also passes through the point (a, b), since these coordinates

satisfy Eq. (3). Thus, if upon the diameters of the circles $\rho = a \cos\theta$ and $\rho = b \sin\theta$, we construct a rectangle, then the circle which has a diagonal of this rectangle as a diameter, is the locus of $\rho = a \cos\theta + b \sin\theta$ (see Fig. 69).

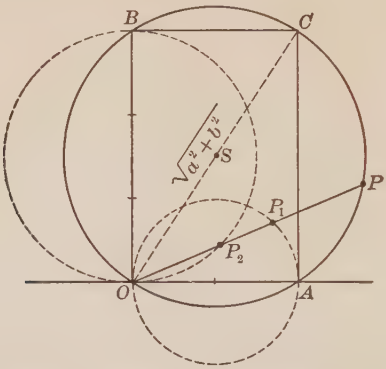


FIG. 69.—The circles $\rho = a \cos\theta$, $\rho = b \sin\theta$, and $\rho = a \cos\theta + b \sin\theta$, or the circles OA, OB, and OC respectively.

Exercises

Draw the graphs for the following:

1. $\rho = 2 \cos\theta + 2 \sin\theta$.
2. $\rho = 3 \cos\theta + 2 \sin\theta$.
3. $\rho = -2 \cos\theta + 2 \sin\theta$.
4. $\rho = -3 \cos\theta - 2 \sin\theta$.

5. In Fig. 70 let $c = 2$ and $\alpha = 30°$. Find the equation of each of the four circles in the form $\rho = a \cos\theta + b \sin\theta$.

74. Composition.—The shearing of a curve in a straight line, considered in Sec. **38,** may be thought of as the addition of the ordinates of the curve and of the straight line, corresponding to a given value of the abscissa. This sum is the corresponding ordinate of the new curve. In the more general case the curve $y = f(x) + F(x)$ may be constructed from the curves $y = f(x)$ and $y = F(x)$ by adding their ordinates. Thus the curve for $y = x^2 + 1/x$ (Fig. 71) was constructed by adding the ordinates of the curves $y = x^2$ and $y = 1/x$. The curve $y = f(x) + F(x)$ is said to be formed *by composition* from the curves $y = f(x)$ and $y = F(x)$. Thus shear is the special type of composition which occurs when one of the component curves is a straight line.

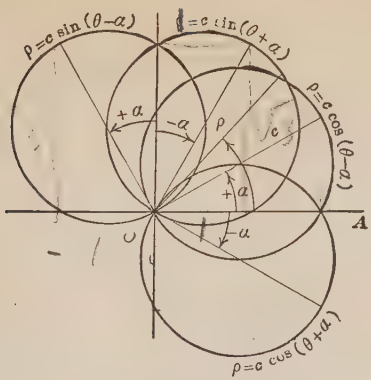


FIG. 70.—Diagram for Exercise 5.

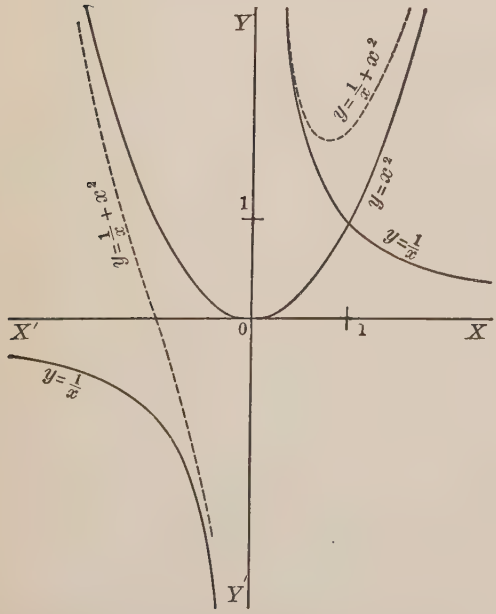


FIG. 71.—The curve $y = 1/x + x^2$ made by composition of the curves $y = 1/x$ and $y = x^2$.

In the same way the curve for $\rho = f(\theta) + F(\theta)$ may be constructed from the curves $\rho = f(\theta)$, and $\rho = F(\theta)$ by adding the radius vectors corresponding to the same value of the vectorial angle. Thus, points on the circle $\rho = 2\cos\theta + 3\sin\theta$ (Fig. 69) may be located by adding (using the bow dividers) the radius vectors of $\rho = 2\cos\theta$ and $\rho = 3\sin\theta$. That is, $OP = OP_1 + OP_2$ for all positions of OP.

Similarly the curve $\rho = a(1 + \cos\theta)$ may be constructed by adding a to every radius vector of the circle $\rho = a\cos\theta$.

Exercises

1. Plot on polar coordinate paper the curve for $\rho = 3\cos\theta + 2\sin\theta$, making use of the circles $\rho = 3\cos\theta$ and $\rho = 2\sin\theta$.

2. Plot on polar coordinate paper the curve for $\rho = \cos\theta + 1$, making use of the circles $\rho = \cos\theta$, and $\rho = 1$. Note that when $90° < \theta < 270°$, the ρ for $\rho = \cos\theta$ is negative, and that the addition referred to above is algebraic addition.

3. Plot upon polar coordinate paper the curve for $\rho = 1 + \sin\theta$, making use of the circles $\rho = 1$, and $\rho = \sin\theta$.

4. Plot upon polar coordinate paper the curve for $\rho = 2\cos\theta - 1$.

5. Plot upon polar coordinate paper the curve for $\rho = \cos\theta + 2$.

6. Construct and compare the curves $\rho = -2 + 3\cos\theta$ and $\rho = 2 - 3\sin\theta$.

7. Construct the curve $\rho = 3 + 5\cos\theta$. For what values of θ does ρ vanish?

8. Between what values does θ range when the inner loop of the curve $\rho = 2 + 5\cos\theta$ is traced?

Miscellaneous Exercises

1. Find the distance of the end of the diameter of

$$\rho = 8\cos(\theta - 60°)$$

from the line OX.

2. If $\rho_1 = a\cos\theta$, and $\rho_2 = a\sin\theta$, find $\rho_1 - \rho_2$ when $\theta = 60°$ and $a = 5$.

3. Find the polar equation of the circle $x^2 + y^2 + 6x = 0$.

4. For what value of θ does $\rho = 3.55$, if $\rho = 2\sin\theta + 3\cos\theta$?

RESULT: $\theta = 23°\ 30'$ and $43°\ 30'$.

Hint: Draw the circles $\rho = 3.55$ and $\rho = 2\sin\theta + 3\cos\theta$ on polar coordinate paper and find the vectorial angles for the two points of

intersection. This problem is the same as: "Solve the equation $2 \sin \theta + 3 \cos \theta = 3.55$ for θ."

5. Solve graphically the equation $2 \sin \theta + 3 \cos \theta = 2$.

Hint: Draw on polar coordinate paper the curves $\rho = 2$ and $\rho = 2 \sin \theta + 3 \cos \theta$.

6. Solve graphically the equation $4 \cos \theta - 3 \sin \theta = 3.5$.

7. Find $\sin \theta$ if $\csc \theta = \dfrac{\sqrt{a^2 + b^2}}{a}$.

8. A circular arc is 4.81 in. long. The radius is 12 in. What angle is subtended by the arc at the center? Give result in radians and in degrees.

9. Certain lake-shore lots are bounded by north and south lines 66 ft. apart. How many feet of lake shore to each lot if the shore-line is straight and runs 77° 30′ east of north?

10. If $y = 2 \sin A + 3 \cos A - 3.55$, take A as 20°, as 23°, as 26°, and find in each case the value of y. From the values of y just found, find a value of A for which y is approximately zero. This process is known as **"cut and try."**

11. The line $y = \frac{3}{2}x$ is to coincide with a diameter of the circle $\rho = 10 \cos (\theta - \alpha)$. Find α.

12. The line $y = 2x$ is to coincide with a diameter of the circle $\rho = 10 \sin (\theta + \alpha)$. Find α.

13. Sketch $y = \frac{1}{2}x$ and $y = \sin x$ and then $y = \frac{1}{2}x - \sin x$.

14. Sketch the curve $y = \cos x + 2 \sin x$, making use of the curves $y = \cos x$ and $y = 2 \sin x$.

15. Find the maximum value of the function given in Exercise 14.

Hint: Find the maximum value of ρ in the graph of $\rho = \cos \theta + 2 \sin \theta$.

16. Find the maximum value of $2 \cos x - 3 \sin x$.

17. Since $\rho = \cos \theta + 2 \sin \theta$ is a circle passing through the pole, the equation may be put in the form $\rho = a \cos (\theta - \alpha)$. Find a and α.

RESULT: $a = \sqrt{5}$ and $\alpha = 63° 20'$ approximately.

18. A circle is inscribed in a 30°, 60° right triangle. Find the diameter of the circle: (*a*) if the shorter leg of the triangle be 2 in.; (*b*) if the longer leg be 2 in.; (*c*) if the hypotenuse be 4 in. (*d*) Find the length of the sides of the triangle if the diameter of the inscribed circle be 2 in.

19. A circle is inscribed in a 45° right triangle. Find the diameter of the circle if the legs of the triangle be 4 in.

20. The center of the circle $\rho = 10 \cos(\theta - \alpha)$ lies on the line $3x - 2y = 1$. Find two possible values for α.

21. The center of the circle $\rho = 10 \sin(\theta + \alpha)$ lies on the line $x - 2y = 6$. Find two possible values for α.

22. Write the Cartesian equations for:

$$(a)\ \rho = 2\cos\theta + 3\sin\theta.$$
$$(b)\ \rho = 2\cos\theta - 5\sin\theta.$$
$$(c)\ \rho = 2\sin\theta - 5\cos\theta.$$

23. Find the coordinates of the center and the radius for:

(a) $x^2 + y^2 - 2x - 4y + 4 = 0$. (d) $2x^2 + 2y^2 + 3x + 5y = 0$.

(b) $x^2 + y^2 + 2x + 4y + 4 = 0$. (e) $3x^2 + 3y^2 - 6x - \sqrt{2}y = 10$.

(c) $x^2 + y^2 + 3x - 4y = 0$. (f) $x^2 + y^2 + 7x - \sqrt{3}y = 25$.

24. Which circles of Exercise 23 pass through the origin?

25. Write the equation of a line passing through the origin and the center of the circle $x^2 + y^2 - 3x - 5y = 6$.

26. Write the equation of a line parallel to $3x - 2y = 6$ and passing through the center of $x^2 + y^2 - 3x - 2y = 0$.

Discuss the following topics from a consideration of the graphs of the functions:

27. In which quadrants is the sine an increasing function of the angle? In which a decreasing function?

28. In which quadrants is the tangent an increasing, and in which a decreasing, function of its variable?

29. In which quadrants are $\cos\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$ increasing and in which are they decreasing functions of θ?

30. Show that all the cofunctions of angles of the first quadrant are decreasing functions.

CHAPTER IV

THE ELLIPSE AND THE HYPERBOLA

75. The Ellipse.—If all ordinates of a circle be shortened by the same fractional amount of their length, the resulting curve is called an **ellipse.** For example, in Fig. 72, the middle points of the positive and the negative ordinates of the large circle were marked and a curve drawn through the points so selected. The result is the ellipse $ABA'B'A$.

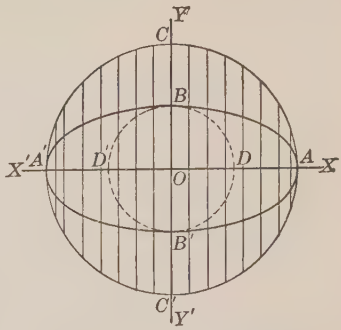


FIG. 72. –Construction of an ellipse.

If

$$x^2 + y^2 = a^2 \qquad (1)$$

is the equation of a circle, then

$$x^2 + (my)^2 = a^2, \qquad (2)$$

in which m is any constant > 1, is the equation of an ellipse, for substituting my for y divides all of the ordinates by m, by Theorem V on Loci, Sec. **29.**

Dividing both members of Eq. (2) by a^2 we obtain

$$\frac{x^2}{a^2} + \frac{m^2}{a^2}y^2 = 1. \qquad (3)$$

Let b^2 be written in place of a^2/m^2. Equation (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (4)$$

which is the standard form of the equation of an ellipse.

If the abscissas of a circle be lengthened by amounts proportional to their lengths, the resulting curve is an ellipse. Let $x^2 + y^2 = b^2$ be the equation of the circle. Then

$$\left(\frac{x}{m}\right)^2 + y^2 = b^2$$

is the equation formed by lengthening all abscissas in the ratio $1:m$, $m > 1$. Dividing by b^2 and replacing m^2b^2 by a^2, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus the ellipse of Fig. 72 could have been formed by doubling all of the abscissas of the circle $BD'B'D$. Hence we see that if all parallel chords of a circle are lengthened or shortened by an amount proportional to their length, an ellipse is formed. If the deformation takes place in chords parallel to either the X- or the Y-axis the equation is of the form of Eq. (4) above, called the **symmetric equation** of the ellipse.

The diameter of the circle from which the ellipse may be formed by shortening parallel chords is called the **major axis** of the ellipse. Thus, AA', or $2a$ (Fig. 72), is the major axis of $ABA'B'$. The diameter of the circle from which the ellipse could have been formed by lengthening parallel chords is called the **minor axis** of the ellipse. Thus, BB', or $2b$ (Fig. 72), is the minor axis of $ABA'B'$. The point of intersection of the axes is called the **center** of the ellipse. One-half of the major and the minor axes are called, respectively, the **semi-major** and **semi-minor** axes of the ellipse. The points A and A' are called the **vertices** of the ellipse.

76. Explicit Form of Equation.—The equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

when solved for y may be put in the important form

$$\mathbf{y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.} \tag{2}$$

The equation of the circle $x^2 + y^2 = a^2$ solved for y is

$$\mathbf{y = \pm \sqrt{a^2 - x^2}.} \tag{3}$$

Equations (2) and (3) are in a form very useful for many purposes. It is easy to see that Eq. (2) states that the ordinates of the ellipse are the fractional amount b/a of the ordinates of the circle (Eq. (3)).

The student will recall that, according to the definitions given in Chap. I, Eq. (1) expresses y as an implicit function of x, while, in Eq. (2), y is an explicit function of x.

77. Orthographic Projection of a Circle.—In Fig. 73 the curve $ABCD$ is a circle which is projected orthographically upon a horizontal plane to produce the curve $A'BC'D$. It is clear from the figure that the half chords, such as MN, of the circle perpendicular to BD are shortened in the ratio 1: cos β to produce the corresponding half chords MN', and hence, by the preceding section, the curve $A'BC'D$ is an ellipse.

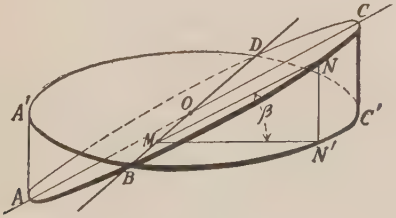


FIG. 73.—If $A'BC'D$ is a circle $ABCD$ is an ellipse. If $ABCD$ is a circle $A'BC'D$ is an ellipse.

78. Section of a Cylinder.—If a circular cylinder be cut by a plane, the section of the cylinder is an ellipse. If, in Fig. 73, we consider the curve $A'BC'D$ to be the circle which is a normal cross-section of the cylinder in question, then the section $ABCD$ is cut from the cylinder by a plane making an angle β with the circular section. Furthermore, the half chords MN are formed from the half chords MN' of the circle by stretching the latter in the ratio sec β : 1. Therefore the section $ABCD$ is an ellipse.

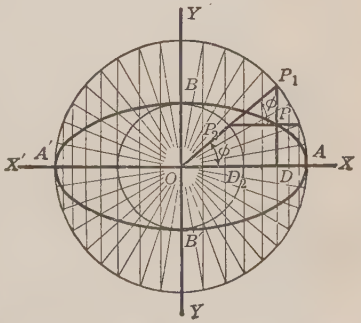


FIG. 74.—A construction of the ellipse.

79. Parametric Equations of the Ellipse.—Let $ABA'B'A$, (Fig. 74) be an ellipse whose semi-major axis is a and whose semiminor axis is b. Upon AA' and BB' as diameters construct circles. These circles are called, respectively, the **major** and the **minor auxiliary circles.** From the origin, draw any radius vector, as OP_2P_1, making an angle ϕ

with the positive direction of the axis of x. Through P_2 and P_1 draw lines parallel, respectively, to the X and the Y-axes, and let P be their point of intersection. It will be shown that P is a point upon the ellipse.

Let the coordinates of P be x and y. Then

$$x = a \cos \phi, \tag{1}$$

and

$$y = b \sin \phi, \tag{2}$$

or

$$\frac{x}{a} = \cos \phi,$$

and

$$\frac{y}{b} = \sin \phi.$$

Then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi + \sin^2 \phi = 1,$$

which shows that P is upon the ellipse.

Equations (1) and (2) are called **parametric equations of the ellipse.** ϕ is called the **variable parameter.**

The method used above of locating points upon the ellipse constitutes one of the best practical methods of constructing an ellipse when its axes are known; for, by it a large number of points upon the ellipse may be easily located and a smooth curve drawn through them.

If the abscissa and the ordinate of any point of a curve are expressed in terms of a third variable, the pair of equations are called the **parametric equations** of the curve, and the third variable is called a **parameter.** The parameter may have a geometrical significance, as in parametric equations of an ellipse, where the parameter has just been seen to be an angle. The variable time, t, is often used as a parameter in terms of which the coordinates x and y of a moving point are expressed.

Thus

$$x = 4t$$
$$y = t + 1.$$

express both x and y as functions of t. If the parameter t be eliminated from these equations, the resulting equation

$$y = \tfrac{1}{4}x + 1$$

is the equation of the straight line which is the path of the moving point. The equations $x = 4t$, $y = t + 1$ are the parametric equations of the straight line

$$y = \tfrac{1}{4}x + 1.$$

Exercises

1. Write the equation of the ellipse formed by diminishing the lengths of all ordinates of $x^2 + y^2 = 4$ by one-half of their length.

2. Write the equation of the ellipse formed by diminishing the lengths of all ordinates of $x^2 + y^2 = 4$ by one-third of their lengths.

3. Write the equation of the ellipse formed by lengthening all ordinates of the circle $x^2 + y^2 = 16$ by one-third of their length.

4. Write the equation of the ellipse formed by lengthening all abscissas of the circle $x^2 + y^2 = 1$ by one-fourth of their length.

5. Write the equation of the ellipse whose semi-axes are 4 and 3.

6. Construct accurately an ellipse whose semi-axes are 3 and 2 in.

7. Construct accurately an ellipse whose parametric equations are $x = 3 \cos \phi$, and $y = 2 \sin \phi$.

8. Write the parametric equations of an ellipse whose semi-axes are 6 and 10.

9. Draw a curve whose parametric equations are $x = \cos \phi$, and $y = \sin \phi$.

10. Find the major and minor axes for the following:

(*a*) $\dfrac{x^2}{81} + \dfrac{y^2}{16} = 1.$ (*c*) $4x^2 + 25y^2 = 100.$

(*b*) $\dfrac{x^2}{16} + \dfrac{y^2}{81} = 1.$ (*d*) $25x^2 + 4y^2 = 100.$

11. Find the axes of the ellipse $k^2x^2 + h^2y^2 = h^2k^2$.

12. Write the equation of the ellipse whose major and minor axes are 10 and 6, respectively.

13. Find the axes of the ellipse whose equation is

$$y = \pm\tfrac{1}{2}\sqrt{36 - x^2}.$$ (Note that a must be 6.)

14. Write the parametric equations of the ellipse

$$y = \pm\tfrac{2}{3}\sqrt{81 - x^2}.$$ (a must be 9; $b = \tfrac{2}{3} \times 9 = 6$.)

15. Discuss the curve

$$x = \pm\tfrac{1}{2}\sqrt{4 - y^2}.$$

16. Discuss the following curves by comparing them with $x^2 + y^2 = 1$:

$$x^2 + 4y^2 = 1.$$
$$4x^2 + y^2 = 1.$$
$$\tfrac{1}{4}x^2 + y^2 = 1.$$

17. Write the Cartesian equation of the curves whose parametric equations are:

(a) $\begin{cases} x = 2\cos\phi. \\ y = \sin\phi. \end{cases}$ (b) $\begin{cases} x = 6\cos\phi. \\ y = 2\sin\phi. \end{cases}$ (c) $\begin{cases} x = \sqrt{3}\cos\phi. \\ y = \sqrt{2}\sin\phi. \end{cases}$

18. What locus is represented by the parametric equations

$$x = 2t + 1$$

and

$$y = 3t + 5?$$

19. What curve is represented by the parametric equations

$$x = 2 + 6\cos\phi$$

and

$$y = 5 + 2\sin\phi?$$

20. Show that the curve

$$x = 3 + 3\cos\phi$$
$$y = 2 + 2\sin\phi$$

is tangent to the coordinate axes.

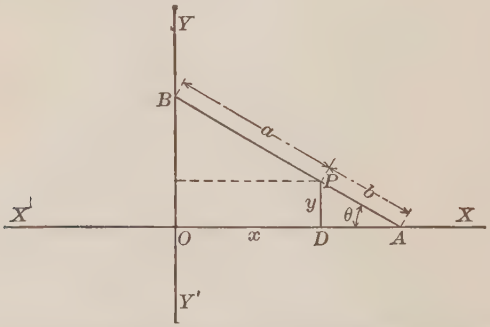


FIG. 75.—A mechanical construction of the ellipse. See Exercise 23.

21. The circle $x^2 + y^2 = 36$ is projected upon a plane. Find an equation of the projection if the angle between the plane and the plane of the circle be 30°.

22. A right circular cylinder is cut by a plane making an angle of 60° with the axis of the cylinder. Find an equation of the curve of intersection, if the radius of the cylinder be 6 units.

23. The line AB (Fig. 75), whose length is $(a + b)$, moves in such a way that the ends A and B always lie on the X- and Y-axes, respectively. Show that the point P describes an ellipse.

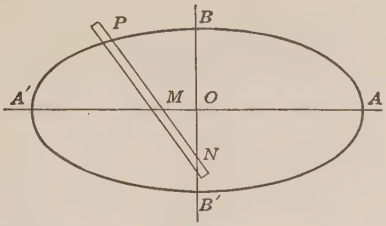


Fig. 76.—Theory of the common "ellipsograph" or elliptic trammel. See Exercise 24.

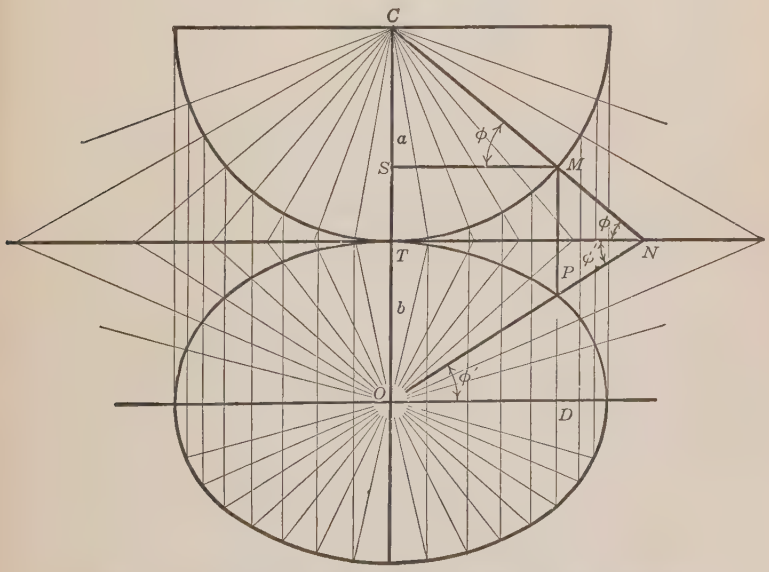


Fig. 77.—A graphical construction of an ellipse. See Exercise 25.

24. The edge of a straight ruler NMP (Fig. 76) is marked so that $PM = b$ and $PN = a$. It is moved, keeping M and N always on AA'

and BB', respectively. Show that P describes an ellipse. The elliptic "trammel" or "ellipsograph" is constructed on this principle by use of adjustable pins on PMN and grooves on AA' and BB'.

25. Draw a semi-circle of radius a about the center C (Fig. 77), and produce a radius to O such that $CTO = a + b$. From C draw any number of lines to the tangent to the circle at T. From O draw lines meeting the tangent at the same points of TN. At the points where the lines from C cut the semi-circle, draw parallels to CT. Show that the points of meeting of the latter with the lines radiating from O determine points on an ellipse, with center at O and semi-axes equal to a and b.

Hint: $OD = SM = a \cos \phi$. From the triangles OPD and ONT

$$\frac{OD}{TN} = \frac{PD}{b}.$$

From the triangles CNT and CMS

$$\frac{SM(=OD)}{TN} = \frac{SC(=a \sin \phi)}{a}.$$

Hence

$$PD = b \sin \phi.$$

80. Origin at a Vertex.—Equation (4), Sec. **75,** Eq. (2), Sec. **76,** and Eqs. (1) and (2), Sec. **79,** are the most useful forms of the equations of the ellipse. It is obvious that the ellipse may be translated by the methods already explained to any position in the plane. The ellipse with center at (h, k) and its axes parallel to the coordinate axes has the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{1}$$

Of special importance is the equation of the ellipse when the origin is taken at the left-hand vertex. This form is best obtained from Eq. (2), Sec. **76,** by translating the curve the distance a in the x direction. Thus,

$$y = \pm \frac{b}{a}\sqrt{a^2 - (x-a)^2},$$

or

$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}x^2,$$

or, letting $2l$ stand for the coefficient of x,

$$y^2 = 2lx - \frac{b^2}{a^2}x^2 = 2lx\left(1 - \frac{x}{2a}\right). \tag{2}$$

For small values of x, $x/2a$ is very small as compared with 1 and the ellipse nearly coincides with the parabola $y^2 = 2lx$.

81. Theorem.—*Any equation of the second degree, lacking the term xy and having the terms containing x^2 and y^2 both present and with coefficients of like signs, represents an ellipse with axes parallel to the coordinate axes.* This is readily shown by putting the equation

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

in the form of Eq. (1) of the preceding section. The procedure is as follows:

$$a\left(x^2 + 2\frac{g}{a}x\right) + b\left(y^2 + 2\frac{f}{b}y\right) = -c. \tag{2}$$

$$a\left(x^2 + 2\frac{g}{a}x + \frac{g^2}{a^2}\right) + b\left(y^2 + 2\frac{f}{b}y + \frac{f^2}{b^2}\right) = \frac{g^2}{a} + \frac{f^2}{b} - c. \tag{3}$$

Let M stand for the expression in the right-hand member of Eq. (3); then we get

$$\frac{\left[x + \frac{g}{a}\right]^2}{\frac{M}{a}} + \frac{\left[y + \frac{f}{b}\right]^2}{\frac{M}{b}} = 1. \tag{4}$$

This shows that Eq. (1) is an ellipse whose center is at the point $(-g/a, -f/b)$ and is constructed from the circles whose centers are at the same point and whose radii are the square roots of the denominators in Eq. (4). The major axis is parallel to OX or OY according as a is less or greater than b. The case when the locus is *not real* should be noted. Compare Sec. **43.**

Illustration: Find the center and the axes of the ellipse

$$x^2 + 4y^2 + 6x - 8y = 23.$$

Write the equation in the form

$$x^2 + 6x + 4y^2 - 8y = 23.$$

Complete the squares

$$x^2 + 6x + 9 + 4y^2 - 8y + 4 = 36.$$

Rewriting

$$(x + 3)^2 + 4(y - 1)^2 = 36.$$

$$\frac{(x + 3)^2}{36} + \frac{(y - 1)^2}{9} = 1.$$

This is seen to be an ellipse whose center is at the point $(-3, 1)$ and whose semi-axes are 6 and 3.

82. The Limiting Lines of an Ellipse.—By the limiting lines of an ellipse are meant those lines, parallel to the X- and Y-axes, which form the rectangle within which the ellipse is described. If the center of the ellipse be at the origin, the equations of these lines are, clearly,

$$x = \pm a,\ y = \pm b;$$

while, if the ellipse be translated a distance x_0 to the right and y_0 up, the limiting lines, which are similarly translated, have for their equations

$$x - x_0 = \pm a,\ y - y_0 = \pm b.$$

Illustration: Find the coordinates of the center, the length of the axes, and the equation of the limiting lines of

$$x^2 + 4x + 9y^2 - 6y = 4.$$

Completing the squares,

$$(x + 2)^2 + 9(y - \tfrac{1}{3})^2 = 9,$$

or

$$\frac{(x + 2)^2}{9} + \frac{(y - \frac{1}{3})^2}{1} = 1.$$

The center of the ellipse is at the point $(-2, \frac{1}{3})$, its semi-axes are 3 and 1. It may be constructed by translating the ellipse $x^2/9 + y^2/1 = 1$, 2 units to the left and $\frac{1}{3}$ unit up. Hence the limiting lines are $x + 2 = \pm 3$ and $y - \frac{1}{3} = \pm 1$, or $x = 1$, $x = -5$, $y = \frac{4}{3}$, and $y = -\frac{2}{3}$.

Exercises

Find the lengths of the semi-axes, the coordinates of the center, and the equations of the limiting lines for the six following loci:

1. $x^2 - 6x + 4y^2 + 8y = 5.$
2. $y^2 - 8y + 4x^2 + 6 = 0.$
3. $12x^2 - 48x + 3y^2 + 6y = 13.$
4. $x^2 + 9y^2 - 12x + 6y = 12.$
5. $4x^2 + y^2 - 12x + 12y - 2 = 0.$
6. $x^2 + 2y^2 - x - \sqrt{2}y = \frac{1}{2}.$
7. Show that $x^2 - 4x + 4y^2 + 8y + 4 = 0$ is an ellipse.
8. Show that $x^2 + 4x + 9y^2 - 6y = 0$ passes through the origin.

9. Discuss the curves:

(*a*) $\frac{x^2}{9}+\frac{y^2}{4}=1$. (*c*) $\frac{x^2}{4}+\frac{y^2}{9}=1$.

(*b*) $\frac{x^2}{9}+\frac{y^2}{4}=-1$. (*d*) $\frac{x^2}{9}+\frac{y^2}{4}=0$.

10. Compare the following parabolas with the standard parabola $y = x^2$ by means of the appropriate Theorems on Loci:

(*a*) $y = 2px^2$. (*c*) $y = \frac{k}{h^2}x^2$.

(*b*) $y = -2px^2$. (*d*) $y = -2px^2 + b$.

What are the roots of the last function?

11. Write the symmetrical equation of the ellipse if its parametric equations are:

$$x = (\tfrac{3}{2})\cos\phi.$$
$$y = (\tfrac{2}{3})\sin\phi.$$

12. Discuss the curve $y^2 = (1\frac{8}{5})x - (\frac{9}{25})x^2$.

13. Find the center of the curve $y^2 = 2x(6 - x)$.

14. Write the parametric equations for the following:

(*a*) $x^2 + 3y^2 = 4$; (*b*) $2x^2 + 5y^2 = 6$; (*c*) $5x^2 + y^2 = 7$.

15. Write the parametric equations for

$$x^2 + 2x + 4y^2 - 16y + 13 = 0.$$

Hint: The equation may be put in the form

$$\frac{(x+1)^2}{4}+\frac{(y-2)^2}{1}=1.$$

Since $x = 2\cos\phi$ and $y = \sin\phi$ are parametric equations for $x^2/4 + y^2/1 = 1$, $x = 2\cos\phi - 1$ and $y = \sin\phi + 2$ are parametric equations of the given ellipse.

16. Write parametric equations for the following: (*a*) $x^2 - 2x + 9y^2 - 6y = 0$; (*b*) $4x^2 + 4x + y^2 - 2y = 5$; (*c*) $x^2 - 4x + y^2 + 6y = 3$.

17. A circle of radius 5 is projected orthographically through an angle of 30°. What is the equation of the resulting ellipse?

18. A circular cylinder of radius 10 is cut by a plane which makes an angle of 45° with a plane normal to the axis of the cylinder. What is the equation of the elliptical section so produced?

83. In Sec. **25** the name rectangular or equilateral hyperbola was given to the graph of the curve $xy = 1$. If both the abscissas and the ordinates of this curve be stretched in the ratio of b to 1, the more general equation $xy = b^2$ results. It is desired to find

the equation of this curve $xy = b^2$ with respect to a new set of axes OX', OY' which make angles of 45° with the original axes OX and OY (see Fig. 78). It is only necessary to express the old coordinates $x = OD$, $y = DP$ in terms of the new coordinates $x' = OD'$, $y' = D'P$ and substitute these values in the equation $xy = b^2$. From the figure

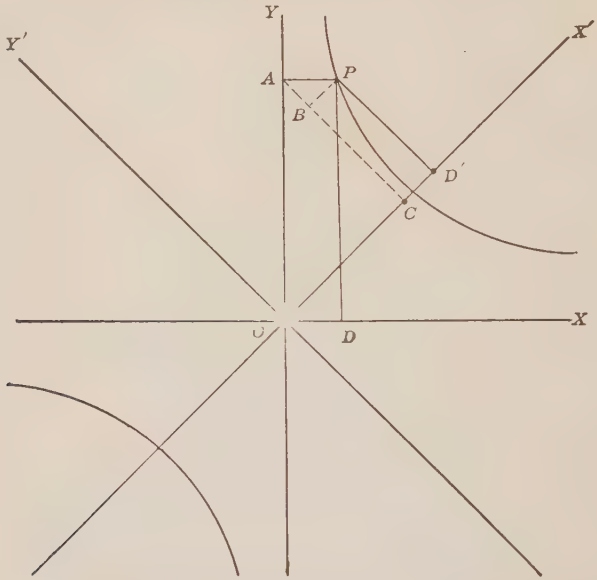


FIG. 78.—The hyperbolas $xy = b^2$ and $x'^2 - y'^2 = 2b^2$.

$$x' = OC + CD' = y \cos 45° + x \cos 45° = \frac{\sqrt{2}}{2}(x + y),$$

$$y' = AC - AB = y \sin 45° - x \sin 45° = \frac{\sqrt{2}}{2}(y - x).$$

Solving these equations for x and y,

$$\begin{aligned} x &= \frac{\sqrt{2}}{2}(x' - y'), \\ y &= \frac{\sqrt{2}}{2}(x' + y'). \end{aligned} \tag{1}$$

Substituting these values in the equation $xy = b^2$, we obtain, as the equation of the curve referred to the new axes,

$$\tfrac{1}{2}(x' - y')(x' + y') = b^2,$$

or

$$x'^2 - y'^2 = 2b^2.$$

If, now, the old axes are discarded, and only the new ones are considered, it is unnecessary to retain the primes, and it is natural to replace the positive constant $2b^2$ by a single positive constant a^2. In Fig. 79 is shown the curve

$$\mathbf{x^2 - y^2 = a^2}. \tag{2}$$

The equation of the asymptotes of the curve $x^2 - y^2 = a^2$ are $y = +x$ and $y = -x$.

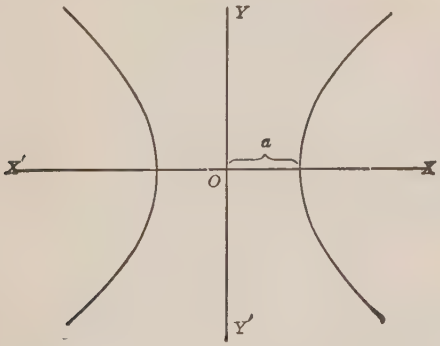


FIG. 79.—The hyperbola $x^2 - y^2 = a^2$.

The curve for $xy = k$ is sometimes called the equilateral hyperbola referred to its asymptotes as axes.

Rotating a set of axes forward 45° is equivalent to rotating a curve backward 45°. Equations (1) were developed without making any use of the fact that P is a point on a hyperbola, and are hence available for use with any curve. Thus, to rotate any locus through an angle $(-45°)$, substitute for x and y the values given by Eqs. (1), dropping the primes after the substitution is made.

84. Parametric Equations.—The parametric equations of the rectangular hyperbola $x^2 - y^2 = a^2$ are

$$\mathbf{x = a \sec \phi}, \tag{1}$$

and

$$\mathbf{y = a \tan \phi}, \tag{2}$$

Dividing Eqs. (1) and (2) by a, squaring, and subtracting,

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = \sec^2 \phi - \tan^2 \phi = 1,$$

which is the same as Eq. (2), Sec. **83.**

Exercises

1. Find the equations of the following curves after rotation about the origin through an angle of $(-45°)$:

$$(a)\ 2xy = 1;\ (b)\ xy = 1;\ (c)\ xy = 4;\ (d)\ xy = \tfrac{8}{2};\ (e)\ xy = 3;$$

$$(f)\ xy - 2 = 0.$$

2. By means of an argument and a figure similar to that above, show that $y^2 - x^2 = a^2$ is the equation of the curve $2xy = a^2$ after rotation about the origin through an angle of $+45°$.

3. Show that $x^2 - y^2 = a^2$ is the equation of the curve $2xy = -a^2$ after rotation about the origin through an angle of $+45°$.

4. Find the equations of the curves given in Exercise 1 after rotation about the origin through an angle of $+45°$.

5. Find the equations of the asymptotes for $x^2 - y^2 - 2x + 4y = 7$.

Hint: Completing the squares,

$$(x - 1)^2 - (y - 2)^2 = 4.$$

Since the asymptotes for $x^2 - y^2 = 4$ are $y = \pm x$, the asymptotes for

$$(x - 1)^2 - (y - 2)^2 = 4 \text{ are } y - 2 = \pm(x - 1),$$

or

$$y = x + 1 \text{ and } y + x = 3.$$

6. Find the equations of the asymptotes and sketch the curves for

$$(a)\ x^2 - y^2 + 2x + 4y = 4.$$

$$(b)\ 2x^2 - 2y^2 + 4x - 8y = 0.$$

85. The Hyperbola of Semi-axes a and b.—The ellipse was defined as the curve produced by lengthening or shortening all ordinates of the circle $x^2 + y^2 = a^2$ an amount proportional to their lengths. Attention has been called to the fact that such a curve results also from the orthographic projection of the circle, or from taking the section of a right circular cylinder by a plane.

From this point on, the name, *an hyperbola*, will, by definition, be applied to any curve obtained from the equilateral hyperbola[1] $x^2 - y^2 = a^2$ by shortening or lengthening all ordinates by an amount proportional to their length. Its equation is then obtained by replacing y in

$$x^2 - y^2 = a^2 \tag{1}$$

by my. Hence we have for the equation of the hyperbola

$$x^2 - (my)^2 = a^2.$$

By dividing by a^2 and replacing a^2/m^2 by b^2, we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{2}$$

the symmetrical form of the equation of the hyperbola. It is easily shown that

$$x = a \sec \phi, \tag{3}$$

and

$$y = b \tan \phi. \tag{4}$$

are parametric equations of the hyperbola whose Cartesian equation is given by Eq. (2). For from Eqs. (3) and (4) we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 \phi - \tan^2 \phi = 1.$$

Note that the parametric equations of the equilateral hyperbola $x = a \sec \phi$ and $y = a \tan \phi$ bear the same relation to the parametric equations of the hyperbola that the parametric equations of the circle bear to the parametric equations of the ellipse.

It is seen that if the ordinates of the asymptotes to the equilateral hyperbola are affected in the same way as the ordinates of the curve itself, *i.e.*, if the asymptotes are considered as part of the locus transformed, they are still the asymptotes to the hyperbola after the transformation.

The equations of the asymptotes of the equilateral hyperbola are

$$y = \pm x.$$

[1] The curves $y = x^n$, $n < 0$, introduced in Sec. **25**, are said to be of hyperbolic type; so that the rectangular hyperbola, or, more generally, an hyperbola, is a curve of hyperbolic type, while a curve of hyperbolic type need not be an hyperbola.

By the transformation they become

$$my = \pm x,$$

or, since

$$m = \frac{a}{b},$$

$$\mathbf{y = \pm \frac{b}{a} x,} \tag{5}$$

which are the equations of the asymptotes of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

86. Construction of the Hyperbola.—To construct the hyperbola draw two concentric circles of radii $OA = a$ and $OB = b$,

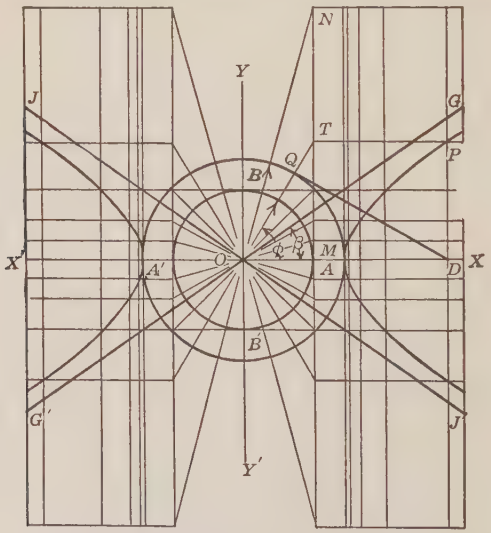


FIG. 80.—The hyperbola $x^2/a^2 - y^2/b^2 = 1$.

as in Fig. 80. Divide each circumference, by means of bow dividers, into the same number of convenient intervals. Lay off, on XOX', distances equal to a sec ϕ by drawing tangents at the points of division on the circumference of the a-circle; also lay off distances equal to b tan ϕ on the vertical tangent to the b-

circle by prolonging the radii of the circle through the points of division of the circumference. Draw horizontal and vertical lines through the points of division of MN and XX', respectively, dividing the plane into a large number of rectangles.

The point of intersection of the vertical and horizontal line corresponding to the same value of ϕ is a point on the hyperbola. The curve may be drawn by starting from the points A and A' and sketching the diagonals of successive rectangles.

In the above construction, there is no reason why the diameter of the b-circle may not be greater than that of the a-circle.

The line $AA' = 2a$ is called the **transverse axis,** the line $BB' = 2b$ is called the **conjugate axis,** the points A and A' are called the **vertices,** and the point O is called the **center** of the hyperbola.

Solving Eq. (2) Sec. **85,** for y, the equation of the hyperbola may be written in the useful form

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}. \tag{1}$$

Compare this equation with the equation of the ellipse (Eq. (2), Sec. **76**).

It is easy to show that the vertical distance PG (Fig. 80) of any point of the curve from the asymptote $G'G$ can be made as small as we please by moving P outward on the curve away from O. Write the equation of the hyperbola in the form

$$y_1 = \frac{b}{a}\sqrt{x^2 - a^2}, \tag{2}$$

and the equation of the asymptote GG' in the form

$$y_2 = \frac{b}{a}x. \tag{3}$$

Then

$$PG = y_2 - y_1 = \frac{b}{a}(x - \sqrt{x^2 - a^2}). \tag{4}$$

Multiply both numerator and denominator in Eq. (4) by $x + \sqrt{x^2 - a^2}$. Then

$$PG = \frac{b}{a}\frac{a^2}{x + \sqrt{x^2 - a^2}}. \tag{5}$$

Now, as x increases in value without limit, the right side of Eq. (5) approaches zero. Whence

$$PG \doteq 0 \text{ as } x \doteq \infty .$$

Exercises

1. Write the symmetrical equation of the hyperbola from the parametric equations $x = 5 \sec \phi$, $y = 3 \tan \phi$.

2. Find the Cartesian equation of the hyperbola from the relations $x = 7 \sec \phi$, $y = 10 \tan \phi$. Note that the graphic construction of the hyperbola holds if $b > a$

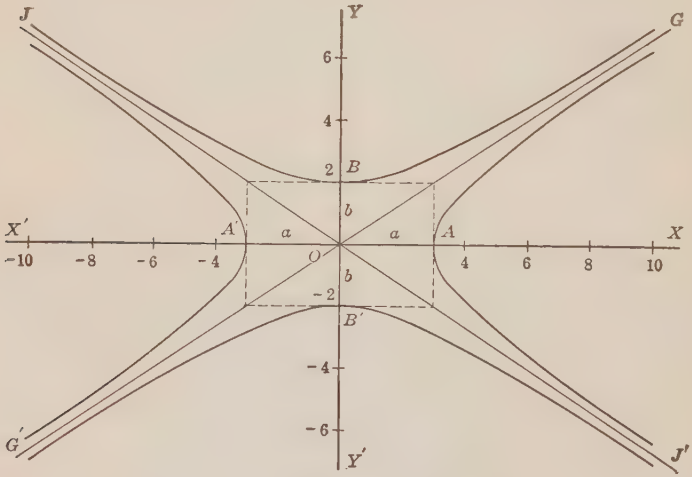


Fig. 81.—Conjugate hyperbolas.

3. What curve is represented by the equation

$$\frac{(x-3)^2}{25} - \frac{(y+2)^2}{16} = 1?$$

4. What curve is represented by the equation $y = \frac{1}{2}\sqrt{x^2 - a^2}$?

5. Write the equation of an hyperbola having the asymptotes $y = \pm(\frac{3}{4})x$, and transverse axis = 24.

6. Show that the curves

$$x^2 + 6x - y^2 - 4y + 4 = 0$$

and

$$(x+3)^2 - (y+2)^2 = 1$$

are the same, and show that each is an hyperbola.

7. What curve is represented by the equations

$$x = h + a \sec \phi$$

and

$$y = k + b \tan \phi?$$

8. Discuss the curve $x^2 - 8x - 2y^2 - 12y = 6$.

87. Conjugate Hyperbolas.—Let $GAJ'JA'G'$ (Fig. 81) be the hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

Its transverse axis is $AA' = 2a$ and its conjugate axis is $BB' = 2b$. Its asymptotes $G'G$ and $J'J$ are the diagonals, produced, of the rectangle constructed upon $A'A$ and $B'B$ as sides.

The hyperbola $GBJG'B'J'$, having $B'B$ as transverse axis and AA' as conjugate axis and $G'G$ and $J'J$ as asymptotes is called the **conjugate** of $GAJ'JA'G'$.

If $Y'OY$ were the X-axis and if $X'OX$ were the Y-axis the equation of the hyperbola $GBJG'B'J'$ would be

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1. \tag{2}$$

By the above supposition we have interchanged x and y. Hence, to get the true equation we must interchange x and y in Eq. (2). Therefore the equation of the hyperbola conjugate to $x^2/a^2 - y^2/b^2 = 1$ is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1. \tag{3}$$

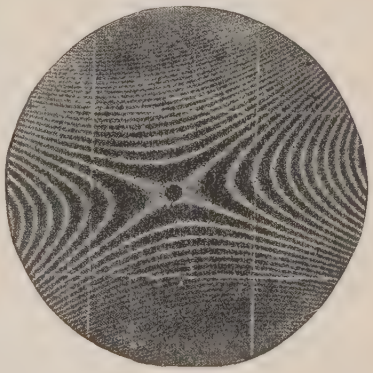


FIG. 82.—A family of conjugate pairs of hyperbolas with common asymptotes. (An interference pattern made from a glass plate under compression. From R. Strauble, "Ueber die Elsticitäts-zahlen und moduln des Glases." Wied. Ann. Bd. 68, 1899, p. 381.)

Figure 82 shows a family of pairs of conjugate hyperbolas.

The student should note that, given an hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the equation of the conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

while the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

which is the equivalent of the two equations $x/a - y/b = 0$, $x/a + y/b = 0$, is the equation of the asymptotes.

Exercises

1. Sketch on the same pair of axes the four following hyperbolas and their asymptotes:

(*a*) $x^2 - y^2 = 25$.

(*b*) $x^2 - y^2 = -25$.

(*c*) $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

(*d*) $\frac{x^2}{25} - \frac{y^2}{9} = -1$.

2. Find the axes of the hyperbola $y = \pm\frac{3}{4}\sqrt{x^2 - 64}$.

3. Sketch the curves:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

and

$$\frac{x^2}{4} - \frac{y^2}{9} = -1.$$

4. Sketch the curves:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

and

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

5. Write the equation of the hyperbola conjugate to

$$y = \pm\tfrac{3}{4}\sqrt{x^2 - 64}.$$

6. Compare the graphs of:

$$y = \pm\tfrac{3}{4}\sqrt{x^2 - 64}.$$
$$y = \pm\tfrac{3}{4}\sqrt{x^2 - 16}.$$
$$y = \pm\tfrac{3}{4}\sqrt{x^2 - 4}.$$
$$y = \pm\tfrac{3}{4}\sqrt{x^2 - 1}.$$
$$y = \pm\tfrac{3}{4}\sqrt{x^2 - \tfrac{1}{16}}.$$
$$y = \pm\tfrac{3}{4}\sqrt{x^2 - 0}.$$

7. Show that $3x^2 - 4y^2 - 7x + 5y + 2 = 0$ is an hyperbola. Find the position of the center and of the vertices. The vertices locate the so-called "limiting lines" of the hyperbola. Write the equations of the asymptotes.

8. Show that $x^2 - 4x - 4y^2 + 4y = 4$ is an hyperbola. Find the coordinates of its center, the equations of its asymptotes, and the equations of its limiting lines.

9. Discuss the graphs:

$$x^2 - y^2 = 1$$

and

$$y^2 - x^2 = 1.$$

10. Discuss the graph $16x^2 - y^2 - 40x - 6y = 2$, and find the limiting lines.

11. Write the equation of the conjugate to

$$\frac{x^2}{4} - \frac{y^2}{16} = 1.$$

12. Write the equation of the conjugate to

$$x^2 - 2x - y^2 - 6y = 24.$$

CHAPTER V

SINGLE AND SIMULTANEOUS EQUATIONS

88. Identities and Conditional Equations.—It is useful to make a distinction between equalities like

$$(a - x)(a + x) = a^2 - x^2, \tag{1}$$

which are true for all values of the variable x, and equalities like

$$x^2 - 2x = 3, \tag{2}$$

which are true only for certain particular values of the unknown number. When two expressions are equal for all values of the variable for which the expressions are defined, the equality is known as an **identity.** When two expressions are equal only for certain particular values of the unknown number, the equality is spoken of as a **conditional equation.** The fundamental formula

$$\sin^2 \phi + \cos^2 \phi = 1$$

is an identity.

$$2 \sin A + 3 \cos A = 3.55$$

is a conditional equation. The symbol $\equiv$ is *sometimes* used to distinguish an identity; thus

$$a^3 - x^3 \equiv (a - x)(a^2 + ax + x^2).$$

The following illustrations and exercises contain problems both in the establishment of trigonometric identities and in the finding of the values of the unknown number from trigonometric conditional equations.

The truth of a trigonometric identity may be established by reducing each side to the same expression. *In this work, however, the student will be required to transform one side by means of the fundamental relations of Eqs.* (2) *to* (6) *Sec.* **56,** *until it is identically equal to the other.*

Facility in the establishment of trigonometric identities is largely a matter of skill in recognizing the fundamental forms, and

of ingenuity in performing transformations. *All solutions of conditional equations should be checked.* The following worked exercises will illustrate the method:

Illustration 1.—Show that $(1 - \sin u \cos u)(\sin u + \cos u) \equiv \sin^3 u + \cos^3 u$. Taking the left-hand member

$$\begin{aligned}(1 - \sin u \cos u)(\sin u + \cos u) &= \sin u + \cos u - \sin^2 u \cos u - \sin u \cos^2 u \\ &= \cos u(1 - \sin^2 u) + \sin u\ (1 - \cos^2 u) \\ &= \cos u \cos^2 u + \sin u \sin^2 u \\ &= \cos^3 u + \sin^3 u.\end{aligned}$$

This last expression is the right-hand member of the given identity. Thus the identity is verified.

Illustration 2.—Show that $\sec^2 x - 1 \equiv \sec^2 x \sin^2 x$.

$$\sec^2 x - 1 = \sec^2 x\left(1 - \frac{1}{\sec^2 x}\right) = \sec^2 x\ (1 - \cos^2 x) = \sec^2 x \sin^2 x$$

Illustration 3.—Solve for all values of x less than 360°

$$2 \sin x + \cos x = 2.$$

Transposing and squaring, we get

$$\cos^2 x = 4 - 8 \sin x + 4 \sin^2 x.$$

Since $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned}1 - \sin^2 x &= 4 - 8 \sin x + 4 \sin^2 x, \\ 5 \sin^2 x - 8 \sin x + 3 &= 0, \\ \sin x &= 1 \text{ or } 0.6\end{aligned}$$

$x = 90°$, and 37° or 143° approximately.

Check:

$$2 \sin 90° + \cos 90° = 2 + 0 = 2.$$

Check:

$$2 \sin 37° + \cos 37° = 1.2 + 0.8 = 2.$$

Does

$$2 \sin 143° + \cos 143° = 1.2 - 0.8 = 0.4 = 2?$$

The last value does not check. The reasons for this will be discussed later in Sec. **94.** Therefore the correct solutions are 90° and 37° approximately.

Exercises

1. Solve $6\cos^2\theta + 5\sin\theta = 7$ for all values of $\theta < 90°$.

SUGGESTION: Write $6(1 - \sin^2\theta) + 5\sin\theta = 7$ and solve the quadratic in $\sin\theta$.

$$6\sin^2\theta - 5\sin\theta + 1 = 0,$$

or

$$(3\sin\theta - 1)(2\sin\theta - 1) = 0.$$

$$\sin\theta = \tfrac{1}{3} \text{ or } \tfrac{1}{2}.$$

$\theta = 19.5°$ approximately and $30°$.

The results should be checked.

2. Prove that for all values of θ (except $\pi/2$ and $3\pi/2$, for which the expressions are not defined)

$$\sec^4\theta - \tan^4\theta \equiv \tan^2\theta + \sec^2\theta.$$

3. Show that

$$\sec^2 u - \sin^2 u \equiv \tan^2 u + \cos^2 u,$$

for all values of the variable u except $90°$ and $270°$, for which the expressions are not defined.

4. Find u, if $\tan u + \cot u = 2$.

5. Find $\sec\theta$, if $2\cos\theta + \sin\theta = 2$.

6. Show that

$$\frac{\sec\alpha + 1}{\tan\alpha} \equiv \frac{\tan\alpha}{\sec\alpha - 1}.$$

Hint: Multiply both numerator and denominator of the left-hand member by $\sec\alpha - 1$.

7. Show that $\sec\alpha + \tan\alpha \equiv \dfrac{1}{\sec\alpha - \tan\alpha}$.

8. Show that $\sin^2\alpha + \sin^2\alpha\tan^2\alpha \equiv \tan^2\alpha$.

9. Show that $(\csc^2\alpha - 1)\sin^2\alpha \equiv \cos^2\alpha$.

10. Show that

$$\frac{\sin A}{1 - \cos A} \equiv \frac{1 + \cos A}{\sin A}.$$

11. Show that $2\cos^2 u - 1 \equiv \cos^4 u - \sin^4 u$.

12. Show that $\cos^4\alpha - \sin^4\alpha + 1 \equiv 2\cos^2\alpha$.

13. Show that $\sec^2 u + \csc^2 u \equiv \csc^2 u \sec^2 u$.

14. Show that $(\tan a + \cot a)^2 \equiv \sec^2 a \csc^2 a$.

15. Solve $\sec x - \tan x + 1 = 0$ for all values of x less than $360°$.

89. Notation of Functions.—It has been pointed out that the symbols $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$, etc. are used to denote functions of x. Likewise, the symbols $f(x, y)$, $F(x, y)$, $\phi(x, y)$, $\psi(x, y)$, etc. are used to represent functions of *two* arguments x and y. For example, $f(x, y)$ in a particular problem may be used to stand for the function $xy/\sqrt{x^2 + y^2}$. We may indicate this fact by writing

$$f(x, y) \equiv \frac{xy}{\sqrt{x^2 + y^2}}. \tag{1}$$

Again, we may abbreviate the function of x and y, $xy \sin x/y$, by the symbol $F(x, y)$. This abbreviation can be indicated by writing

$$F(x, y) \equiv xy \sin \frac{x}{y}. \tag{2}$$

The equation

$$F(x, y) = 0 \tag{3}$$

indicates that y is a function of x; y is a function of x expressed **implicitly.** If Eq. (3) were solved for y, giving

$$y = f(x), \tag{4}$$

y is a function of x expressed **explicitly.** Equations (3) and (4) represent the same functional relation between x and y. Thus $x^2 + y^2 - a^2 = 0$ shows that y is a function of x but the functional relation is expressed implicitly. If the equation be solved for y, giving $y = \pm\sqrt{a^2 - x^2}$, the same functional relation between x and y holds, but now y is an *explicit* function of x.

In the same problem or discussion the symbols $f(x)$, $f(y)$, $f(u)$, or $f(v)$ denote the same functional form, although the arguments may differ. If

$$f(x) \equiv \frac{x}{\sqrt{x^2 + 1}},$$

$f(y)$ means the same function but with every x replaced by y, thus

$$f(y) \equiv \frac{y}{\sqrt{y^2 + 1}}.$$

Again, if in a particular problem or discussion

$$f(x) \equiv x^2 + 2x - 1,$$

then

$$\begin{aligned} f(y) &\equiv y^2 + 2y - 1, \\ f(2) &\equiv 2^2 + 2 \cdot 2 - 1 = 7, \\ f(-1) &\equiv (-1)^2 + 2(-1) - 1 = -2, \\ f(0) &\equiv 0 + 0 - 1 = -1. \end{aligned}$$

Exercises

1. If $f(x) \equiv x^2 + 3x + 2$, find $f(y)$; $f(3)$; $f(0)$; $f(-1)$; and $f(-2)$.

2. If $f(x) \equiv x^3 + 2x^2 + x$, find $f(-1)$; $f(0)$; $f(+1)$; $f(z)$; $f(1/y)$; and $f(t^2)$.

3. If $F(\theta) \equiv \sin\theta$, find $F(\pi/2)$; $F(\pi)$; $F(0)$; $F(\pi/6)$; $F(\pi/3)$ and $F(\frac{3}{2}\pi)$.

4. If $\varphi(\theta) \equiv \tan\theta$, find $\varphi(0)$; $\varphi(\pi/6)$; $\varphi(\pi/3)$; $\varphi(\pi/2)$; $\varphi(\pi)$ and $\varphi(\frac{3}{4}\pi)$.

5. If $f(x, y) \equiv xy/\sqrt{x^2 + y^2}$, find $f(2, 1)$; $f(0, 2)$; $f(z, t)$ and $f(m, n)$.

Hint: To find $f(2, 1)$ replace x by 2, and y by 1 in the given function of x and y.

90. A polynomial in x of the nth degree is defined as

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ . \ . \ . \ + a_{n-1}x + a_n,$$

where the symbols a_0, a_1, a_2, . . . stand for any real constants whatsoever, positive or negative, integral or fractional, rational or irrational, and where n is any positive integer. If none of the coefficients be zero, the number of terms in a polynomial in x of the nth degree is $n + 1$.

In what follows in this chapter $f(x)$ is supposed to stand for a polynomial in x.

91. The Remainder Theorem.—Let

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ . \ . \ . \ + a_{n-1}x + a_n. \qquad (1)$$

Then

$$f(r) \equiv a_0r^n + a_1r^{n-1} + a_2r^{n-2} + \ . \ . \ . \ + a_{n-1}r + a_n. \qquad (2)$$

By subtracting Eq. (2) from Eq. (1),

$$f(x) - f(r) \equiv a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \ . \ . \ . \ + a_{n-1}(x - r). \qquad (3)$$

The right-hand side of this equation is made up of a series of terms containing differences of like powers of x and r, and, hence, by the

well-known theorem in factoring,[1] each binomial term is exactly divisible by $x - r$. The quotient of the right-hand side of Eq. (3) by $x - r$ may be written out at length, but it is sufficient to abbreviate it by the symbol $Q(x)$ and write

$$\frac{f(x) - f(r)}{x - r} = Q(x), \tag{4}$$

or

$$\frac{f(x)}{x - r} = Q(x) + \frac{f(r)}{x - r}. \tag{5}$$

Equation (5) shows that $f(r)$ is the remainder when $f(x)$ is divided by $x - r$. Thus we have the **Remainder Theorem:**

If a polynomial in x be divided by $x - r$ the remainder which does not contain x is obtained by writing, in the given function, r in place of x. This theorem shows, for example, that the remainder of the division

$(x^3 - 6x^2 + 11x - 6) \div (x - 4)$ is $4^3 - 6(4)^2 + 11(4) - 6$, or 6; also that the remainder of the division

$$(x^3 - 6x^2 + 11x - 6) \div (x + 1)$$

is

$$(-1)^3 - 6(-1)^2 + 11(-1) - 6 = -24.$$

The theorem enables one to write the remainder without actually performing the division.

Exercises

Without performing the division, find the remainder of the following divisions:

1. $(x^2 + 3x - 2) \div (x - 1)$.

2. $(x^3 + 3x^2 + 2x - 1) \div (x - 2)$.

3. $(x^4 + 4x^3 + 3x^2 - 6x - 1) \div (x + 1)$.

4. $(x^3 - 3x^2 + 2x - 1) \div (x - 2)$.

5. $(x^2 + 3x + 2) \div (x + 1)$.

6. $(x^2 + 3x + 2) \div (x + 2)$.

92. Factor Theorem.—From Eq. (5) of the preceding section, we see that if $f(r)$ be zero, the remainder of the division of $f(x)$ by $x - r$ is zero, or $f(x)$ is exactly divisible by $x - r$, *i.e.*, $x - r$ is a factor of $f(x)$. Thus we have the **Factor Theorem:**

[1] See Appendix, Chap. XV, Sec. 306.

If a polynomial in x becomes zero when r is written in the place of x, $x - r$ is a factor of the polynomial. This means, for example, that if 3 be substituted for x in the function $x^3 - 6x^2 + 11x - 6$ and if the result $3^3 - 6(3)^2 + 11(3) - 6$ is zero, then $x - 3$ is a factor of $x^3 - 6x^2 + 11x - 6$.

The value r of the argument x that causes the function to take on the value zero has already been named a **root** or a **zero** of the function. The factor theorem may, therefore, be stated in the form: *A polynomial in x is exactly divisible by $x - r$ where r is any root of the polynomial.*

The familiar method of solving a quadratic equation by factoring is nothing but a special case of the present theorem. Thus, if

$$x^2 - 5x + 6 = 0,$$
$$(x - 2)(x - 3) = 0;$$

and the roots are $x = 2$ and $x = 3$. The numbers 2 and 3 are such that when substituted in $x^2 - 5x + 6$ the expression is zero; and the factors of the expression are $x - 2$ and $x - 3$ by the factor theorem.

Exercises

1. Tabulating the cubic polynomial $f(x) \equiv x^3 - 6x^2 + 11x - 6$, we obtain:

x	-3	-2	-1	-0	1	1.5	2	2.5	3	4
$f(x)$,	-120,	-60,	-24,	-6,	0,	$+0.375$,	0,	-0.375,	0,	6

What is the remainder when the function is divided by $x - 4$? By $x + 2$? By $x + 3$? By $x - 1.5$? By $x - 3$?

Name three factors of the above function.

2. Find the remainder when $x^4 - 5x^3 + 12x^2 + 4x - 8$ is divided by $x - 2$.

3. Show by the remainder theorem that $x^n + a^n$ is divisible by $x + a$ when n is an odd integer, but that the remainder is $2a^n$ when n is an even integer.

4. Without actual division, show that $x^4 - 4x^2 - 7x - 24$ is divisible by $x - 3$.

5. Show that $a^4 + a^2 - ab^3 - b^2$ is divisible by $a - b$.

6. Show that $(x + 1)^2(x - 2) - 4(x - 1)(x - 5) + 4$ is divisible by $x - 1$.

7. Show that $6x^5 - 3x^4 - 5x^3 + 5x^2 - 2x - 3$ is divisible by $x + 1$.

8. Show that $(b - c)(b + c)^2 + (c - a)(c + a)^2 + (a - b)(a + b)^2$ is divisible by $(b - c)(c - a)(a - b)$.

Hint: First consider the function as a polynomial in b; then as a polynomial in c; and then as a polynomial in a.

9. Show that $(b - c)^3 + (c - a)^3 + (a - b)^3$ is divisible by $(b - c)(c - a)(a - b)$.

93. An Equation with Given Roots.—The factor theorem enables us to build up a polynomial having given roots. If, for example, 1, 2, and 3 are roots, $x - 1$, $x - 2$, and $x - 3$ are factors of the polynomial. Hence $(x - 1)(x - 2)(x - 3)$, or $x^3 - 6x^2 + 11x - 6$ is a factor of the polynomial. Introducing another factor k, which does not contain x, cannot introduce another root, as α, for k cannot contain the factor $x - \alpha$.

For the same reason, multiplying the equation $x^3 - 6x^2 + 11x - 6 = 0$ by k, when k does not contain x, cannot introduce roots, or solutions, in the equation. On the other hand, if the equation be multiplied by a function of x, roots of the equation may be introduced or removed, for, clearly, if the multiplier contain the factor $x - \alpha$, the root α will be introduced; and if the multiplier contain the factor $x - 1$ in its denominator, the factor $x - 1$ will be divided out from both numerator and denominator, if it be a factor of the numerator, and the root 1 will be removed from function.

Exercises

Build up polynomial equations having the following numbers for roots:

1. 1, 3, and 4.
2. -1, 2, and -3.
3. 0, 2, and -1.
4. 1, 0, 0, and 2.

94. Legitimate and Questionable Transformations.—If one equation is derived from another by an operation which has no effect one way or another on the solution, it is spoken of as a **legitimate transformation**; if the operation is of such a nature that it *may* have an effect upon the roots, it is called a **questionable transformation,** meaning thereby that the effect of the operation requires examination.

In performing operations on the members of equations, the effect on the solution must be noted, and proper allowance made in the result. It cannot be too strongly emphasized that *the test for any solution of an equation is that it satisfy the original equation.* "No matter how elaborate or ingenious the process by which the solution has been obtained, if it do not stand this test it is no solution; and, on the other hand, no matter how simply obtained, provided it do stand this test, it is a solution."[1]

By the principles or axioms of algebra, an *equation remains true* if we unite the same number to both sides by addition or subtraction; or if we multiply or divide both members by the same number, *not zero;* or if like powers or roots of both members be taken. But we have indicated in the preceding section that these operations *may affect the number of roots of the equation* This is obvious enough in the case already cited. Sometimes, however, the operation that removes or introduces roots is so natural and its effect is so disguised that the student is apt not to take due account of its effect. Thus, the roots of

$$3(x - 5) = x(x - 5) + x^2 - 25 \qquad (1)$$

are -1 and 5, for either of these when substituted for x will satisfy the equation. Dividing the equation through by $x - 5$, the resulting equation is

$$3 = x + x + 5.$$

This equation is not satisfied by $x = 5$. One root has disappeared in the transformation. It is easy to keep account of this if Eq. (1) be given in the form

$$(x - 5)(x + 1) = 0,$$

but the fact that a factor has been removed may be overlooked when the equation is written in the form first given.

A very important effect upon the roots of an equation results from squaring both members. The student must always take proper account of the effect of this common operation. To illustrate, take the equation

$$x + 5 = 1 - 2x. \qquad (2)$$

[1] Chrystal's "Algebra."

It is satisfied only by the value $x = -4\frac{1}{3}$. Now, by squaring both sides of the equation, we obtain

$$x^2 + 10x + 25 = 1 - 4x + 4x^2,$$

which is satisfied by either $x = 6$ or $x = -\frac{4}{9}$. Here, obviously, an extraneous solution has been introduced by the operation of squaring both members.

It is easy to show that squaring both members of an equation is equivalent to multiplying both sides by the sum of the left and the right members. Thus, let any equation be represented by

$$L(x) = R(x), \tag{3}$$

in which $L(x)$ represents the given function of x that stands on the left-hand side of the equation and $R(x)$ represents the given function of x that stands on the right-hand side of the equation. Squaring both sides,

$$[L(x)]^2 = [R(x)]^2.$$

Transposing,

$$[L(x)]^2 - [R(x)]^2 = 0,$$

factoring,

$$[L(x) + R(x)]\,[L(x) - R(x)] = 0.$$

But Eq. (3) may be written

$$L(x) - R(x) = 0.$$

Thus, by squaring the members of Eq. (3) the factor $L(x) + R(x)$ has been introduced.

The sum of the left- and the right-hand members of Eq. (2), above, is $6 - x$. Hence, squaring both sides of Eq. (2) is equivalent to the introduction of this factor, or the operation introduces the root 6, as already noted.

As another example, suppose that it be required to solve

$$\sin \alpha \cos \alpha = \tfrac{1}{4} \tag{4}$$

for $\alpha < 90°$. Substituting for $\cos \alpha$, Eq. (4) becomes

$$\sin \alpha \sqrt{1 - \sin^2 \alpha} = \tfrac{1}{4}, \tag{5}$$

squaring

$$\sin^2 \alpha (1 - \sin^2 \alpha) = \tfrac{1}{16},$$

completing the square

$$\sin^4 \alpha - \sin^2 \alpha + \tfrac{1}{4} = \tfrac{3}{16}.$$

Hence,

$$\sin \alpha = \pm\sqrt{\tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{3}}$$
$$= \pm 0.9659 \text{ or } \pm 0.2588.$$

Only the positive values satisfy Eq. (4); the negative values were introduced in squaring Eq. (5). If, however, *the restriction* $\alpha < 90°$ *be removed, so that the radical in Eq.* (5) *must be written with the double sign, then no new solutions are introduced by squaring.*

Among the common operations that have no effect on the solution are multiplication or division by known numbers, or addition or subtraction of like terms to both members; none of these introduce factors containing the unknown number. Taking the square root of both numbers is legitimate *if the double sign be given to the radical.* Clearing of fractions is legitimate if it be done so as not to introduce a new factor. If the fractions are not in their lowest terms, or if the equation be multiplied through by an expression having more factors than the least common multiple of the denominators, new solutions *may* appear, for extra factors are probably thereby introduced. Hence, in clearing of fractions, the multiplier should be the least common denominator and the fractions should be in their lowest terms. This, however, does not constitute a *sufficient condition,* therefore *the only certainty lies in checking all results.*

Exercises

Suggestions: It is important to know that any equation of the form

$$ax^{2n} + bx^n + c = 0$$

can be solved as a quadratic by finding the two values of x^n. Frequently equations of this type appear in the form

$$dx^n + ex^{-n} = f.$$

Likewise, any equation of the form

$$af(x) + b\sqrt{f(x)} + c = 0$$

can be solved as a quadratic by finding the two values of $\sqrt{f(x)}$ and then solving the two equations resulting from putting $\sqrt{f(x)}$ equal to each of them. One of these usually gives extraneous solutions.

These two types occur in the exercises given below.

Since operations which introduce extraneous solutions are often used in solving equations, the only sure test for the solution of any equation is to check the results by substituting them in the original equation.

Take account of all questionable operations in solving the following equations:

1. $\dfrac{3x}{x-3} = \dfrac{6}{x+3} + \dfrac{9}{x-3}$. NOTE: 3 is *not* a root.

2. $\left(\dfrac{x^2+5x+6}{(x-3)}\right) + 4x - 7 = -15$.

3. $3(x-5)(x-1)(x-2) = (x-5)(x+2)(x+3)$.

NOTE: Divide by $(x-5)$ but take account of its effect.

4. $\dfrac{x^2}{a} + ax = \dfrac{x^2}{b} + bx$.

5. $ax(cx-3b) = 5a(3b-cx)$.

6. $x^2 - n^2 = n - x$.

7. $(x-4)^3 + (x-5)^3 = 31[(x-4)^2 - (x-5)^2]$. Divide by $(x-4) + (x-5)$ or $2x-9$.

8. $\dfrac{x^2-3x}{x^2-1} + 2 + \dfrac{1}{x-1} = 0$. If the fractions be added, multiplication is unnecessary. There is only one root.

9. $x = 7 - \sqrt{x^2-7}$.

10. $\sqrt{x+20} - \sqrt{x-1} - 3 = 0$.

11. $\sqrt{\dfrac{15}{4+x}} = \dfrac{3}{2} + \sqrt{x}$.

12. $\dfrac{20x}{\sqrt{10x-9}} - \sqrt{10x-9} = \dfrac{18}{\sqrt{10x-9}} + 9$.

13. $\dfrac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}-\sqrt{x-3}} = \dfrac{3}{x-3}$. Consider as a proportion and take by composition and division.

14. $x^{\frac{2}{3}} + \dfrac{5}{2} = \left(\dfrac{13}{4}\right)x^{\frac{1}{3}}$.

15. $\sqrt[4]{x^3} - 2\sqrt{x} + x = 0$. Divide by $\sqrt{x}$.

16. $2\sqrt{x^2-5x+2} - x^2 + 8x = 3x - 6$. Call $x^2 - 5x + 2 = u^2$.

17. $4x^2 - 4x + 20\sqrt{2x^2-5x+6} = 6x + 66$.

18. $x^{-2} - 2x^{-1} = 8$.

19. $x^{\frac{2}{11}} - 5x^{\frac{1}{11}} + 4 = 0$.

20. $110x^{-4} + 1 = 21x^{-2}$.

21. $\sqrt{x} + 4x^{-\frac{1}{2}} = 5$.

22. $8x^{\frac{3}{2}} - 8x^{-\frac{3}{2}} = 63$.

23. $(x-a)^n - 3(x-a)^{-n} = 2$.

24. $2x^{\frac{1}{3}} - 3x^{\frac{2}{3}} + x = 0$.

95. Intersection of Loci.—In Secs. **20** and **41** we have seen that the coordinates of the points of intersection of two loci are to be obtained by solving simultaneously the equations of the loci. We are now in a position to trace the graphic significance of the various steps in this process of solution, and to observe at just what point extraneous roots are introduced. For example, to find the points of intersection of the circle $x^2 + y^2 = 25$ and the straight line $x + y = 7$ we solve the equations by the usual method, as follows:

$$\left.\begin{aligned} x^2 + y^2 &= 25 \\ x + y &= 7 \end{aligned}\right\}. \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

The graphs are a circle and a straight line, as shown in (1) (Fig. 83). Squaring the second equation, the system becomes

$$\left.\begin{aligned} x^2 + y^2 &= 25 \\ x^2 + 2xy + y^2 &= 49 \end{aligned}\right\}. \qquad \begin{matrix}(1)\\(3)\end{matrix}$$

The second equation is now equivalent to the two equations $x + y = +7$ and $x + y = -7$ and, therefore, represents the *two* straight lines shown in (2) (Fig. 83). The effect of squaring has been to introduce two extraneous solutions corresponding to the points P_3 and P_4, for Eq. (3) may be written $(x + y + 7)(x + y - 7) = 0$, while Eq. (2) from which it was derived, is $x + y - 7 = 0$.

Multiplying Eq. (1) by 2 and subtracting Eq. (3) from it, the last pair of equations becomes

$$\left.\begin{aligned} x^2 - 2xy + y^2 &= 1 \\ x^2 + 2xy + y^2 &= 49 \end{aligned}\right\}, \qquad \begin{matrix}(4)\\(3)\end{matrix}$$

which gives the four straight lines of Fig. 89, (3). Taking the square root of each member, but discarding the equation $x + y + 7 = 0$, because it corresponds to the extraneous solutions introduced by the questionable operation, we have:

$$\left.\begin{aligned} x - y &= \pm 1 \\ x + y &= 7 \end{aligned}\right\}. \qquad \begin{matrix}(5)\\(2)\end{matrix}$$

By addition and subtraction we obtain the results:

$$\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}, \qquad (6)$$

$$\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}, \qquad (7)$$

represented by the intersections of the lines parallel to the axes shown in Fig. 83, (4).

This is a good illustration of the graphic changes that take place during the solution of simultaneous equations of the second degree. The ordinary algebraic solution consists, geometrically, in the successive replacement of loci by others of an entirely different kind,

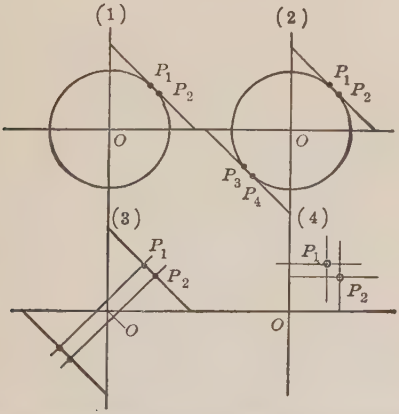


FIG. 83.—Graphic representation of the steps in the solution of a certain set of simultaneous equations.

but all passing through the points of intersection (as P_1, P_2, Fig. 83) of the original loci. The final loci are straight lines parallel to the axes.

Exercises

Find the coordinates of the points of intersection of the following pairs of equations; sketch curves representing all equations involved in the solution:

1. $xy = 1.$
$3x - 5y = 2.$

2. $x^2 + y^2 = 5.$
$y^2 = 4x.$

3. $x^2 + x = 4y^2.$
$3x + 6y = 1.$

4. $x^2 + y^2 = 9.$
$x^2 - y^2 = 4.$

Hint for Example 4: Let $u = x^2$ and $v = y^2$. Solve for u and v.

Solve graphically the following:

5. $x^2 + y^2 = 25.$
$x + y = 2.$

6. $x^2 + y^2 = 25.$
$x^2 + y^2 + 2x - 6y + 6 = 0.$

7. $y = x^2 + x - 1.$
$xy = 1.$

8. Show that

$$x^2 + y^2 = 25$$
$$x + y = 1$$

has a solution, but there is no real solution of the system

$$x^2 + y^2 = 25$$
$$x + y = 11.$$

9. Do the curves

$$x^2 + y^2 = 25$$
$$xy = 100$$

intersect?

10. Do the curves

$$x^2 + y^2 = 25$$
$$xy = 12$$

intersect?

11. Solve

$$x + y = 6.$$
$$xy = 5.$$

12. Find the coordinates of the points of intersection of the curves

$$x^2 + y^2 = 34.$$
$$xy = 15.$$

The circle and the hyperbola represented by these equations should be drawn by the student.

96. Graphic Solution of the Cubic Equation.—The roots of a cubic $x^3 + \alpha x^2 + \beta x + \gamma = 0$ (where α, β, and γ are given known numbers) may be determined graphically as explained in Sec. **40.**

Another method of solving the cubic equation graphically will now be given. The roots of the equation

$$x^3 + \alpha x^2 + \beta x + \gamma = 0 \tag{1}$$

are the X-intercepts for the graph of

$$y = x^3 + \alpha x^2 + \beta x + \gamma, \tag{2}$$

If we replace x in Eq. (2) by $(x - k)$, where k is a constant, Eq. (2) becomes

$$y = (x - k)^3 + \alpha(x - k)^2 + \beta(x - k) + \gamma,$$

or

$$y = x^3 + (\alpha - 3k)x^2 + (\beta - 2\alpha k + 3k^2)x - (k^3 - \alpha k^2 + \beta k - \gamma). \quad (3)$$

It will be noticed that if k be chosen equal to $\alpha/3$ the coefficient of x^2 vanishes and Eq. (3) becomes

$$y = x^3 + ax + b, \quad (4)$$

where a stands for the coefficient of x and b stands for the absolute term of Eq. (3).

Since the graph for Eq. (4) differs from the graph of Eq. (2) only in that it is translated $\alpha/3$ units parallel to the X-axis, the X-intercepts for the first graph are $\alpha/3$ units greater (less if α be negative) than the X-intercepts for the second graph. Hence, if the roots of

$$x^3 + ax + b \quad (5)$$

can be found, these roots decreased by $\alpha/3$ are the roots of Eq. (1).

Since an equation of the form of Eq. (2) can always be put in the form of Eq. (4), we shall only consider cubic equations of the form

$$x^3 + ax + b = 0. \quad (6)$$

Consider the system of curves

$$y = x^3. \quad (7)$$

$$y = -ax - b. \quad (8)$$

Equation (7) gives the cubic parabola, and Eq. (8) the straight line (Fig. 84).

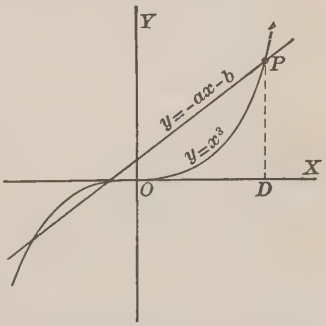


FIG. 84. — Construction for graphical solution of $x^3 + ax + b = 0$.

Let P be a point of intersection of the cubic and the straight line. Let OD be the abscissa of the point P. The value of OD is a root of Eq. (6), for OD is a value of x for which $x^3 = -ax - b$, or for which $x^3 + ax + b = 0$.

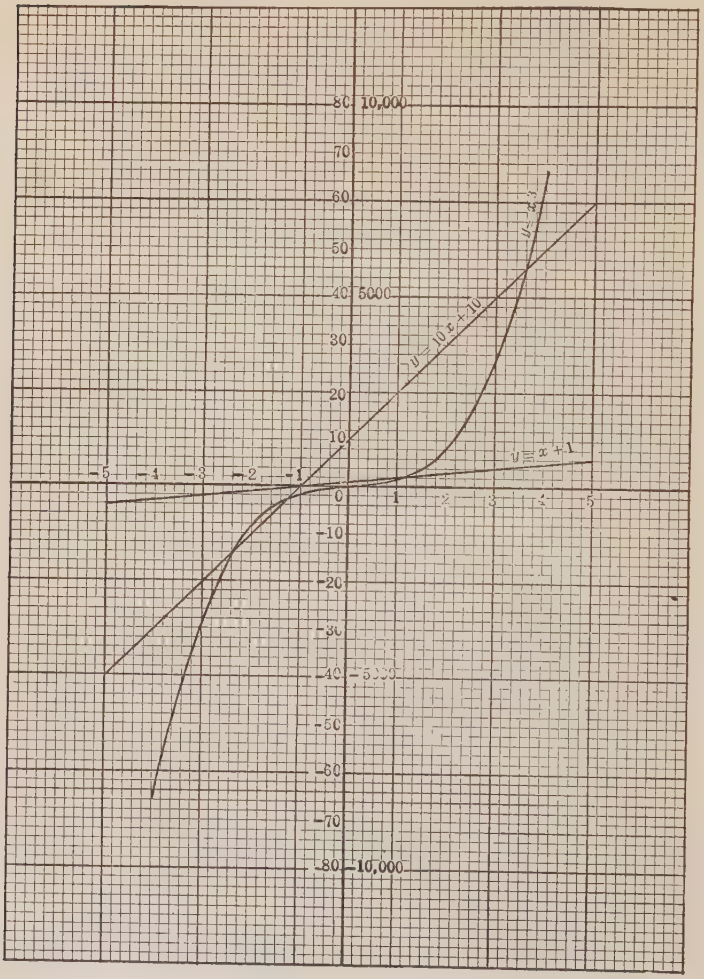


FIG. 85.—Graphic solution of the cubic $x^3 - x - 1 = 0$ and $x^3 - 10x - 10 = 0$.

In drawing the graph of the cubic parabola, it is desirable to use, for the y-scale, one-tenth of the unit used for the x-scale, so as to bring a greater range of values for y upon an ordinary sheet of coordinate paper. The cubic parabola graphed to this scale is shown in Fig. 85. The diagram gives the solution of $x^3 - x - 1 = 0$. The graphs $y = x^3$ and $y = x + 1$ are seen to intersect at $x = 1.32$. This, then, should be one root of the cubic correct to two decimal places. The line $y = x + 1$ cuts the cubic parabola in but one point, which shows that there is but one real root of the cubic. To obtain the imaginary roots, divide $x^3 - x - 1$ by $x - 1.32$. The result of the division, retaining but two places of decimals in the coefficients, is

$$x^2 + 1.32x + 0.7424.$$

Putting this equal to zero and solving by completing the square, we find

$$x = -0.66 \pm 0.55\sqrt{-1},$$

in which, of course, the coefficients are not correct to more than two places.

The equation

$$x^3 - 10x - 10 = 0 \tag{9}$$

illustrates a case in which the cubic has three real roots. The straight line $y = 10x + 10$ cuts the cubic parabola (see Fig. 85) at $x = -1.2$, $x = -2.4$, and $x = 3.6$. These, then, are the approximate roots. The product

$$(x + 1.2)(x + 2.4)(x - 3.6) = x^3 - 10.08x - 10.37$$

should give the original Eq. (9). This result checks the work to about two decimal places.

The x-scale of Fig. 85 extends only from -5 to $+5$. The same diagram may, however, be used for any range of values by suitably changing the unit of measure on the two scales; thus, the divisions of the x-scale may be marked with numbers fivefold the present numbers, in which case the numbers on the y-scale must be marked with numbers 125 times as great as the present numbers. These results are shown by the auxiliary numbers attached to the y-scale in Fig. 85.[1]

It is obvious that a similar process will apply to any equation of the form

$$x^n + ax + b = 0.$$

[1] For other graphic methods of solution of equations, see Runge's "Graphical Methods," Columbia University Press, 1912.

Exercises

Solve graphically the following equations and check each result separately:

1. $x^3 - 4x + 10 = 0$.
2. $x^3 - 12x - 8 = 0$.
3. $x^3 + x - 3 = 0$.
4. $x^3 - 15x - 5 = 0$.
5. $x^3 - 3x + 1 = 0$.
6. $x^3 - 4x - 2 = 0$.
7. $2 \sin \theta + 3 \cos \theta = 3.5$.

NOTE: Construct on polar paper the circles $\rho = 3.5$ and $\rho = 2 \sin \theta + 3 \cos \theta$.

8. $2x + \sin x = 0.6$.

NOTE: Find the intersection of $y = \sin x$ and the line $y = -2x + 0.6$. If 1.15 in. is the amplitude of $y = \sin x$, then 1.15 in. must be the unit of measure used for the construction of the line $y = -2x + 0.6$.

9. $x^3 + x + 1 + 1/x = 0$.
10. Show that $x^3 + ax + b = 0$ can have but one real root if $a > 0$.

97. Method of Successive Approximations.—It must be remembered that the graphic methods of solving numerical equations by finding one or both coordinates of points of intersection of graphs gives results only approximately correct. The degree of accuracy depends upon the scale of the drawing and upon the accuracy with which the graphs are constructed. The results thus obtained may be used as a first approximation to the solution by a method illustrated below:

Suppose that it be required to find to four decimal places one root of $x^3 - x - 1 = 0$ (see Sec. **96** and Fig. 85). The graphic method gives $x = 1.32$. This is the first approximation. A second approximation is found as follows:

Substituting 1.32 for x in

$$y = x^3 - x - 1 \tag{1}$$

gives -0.0200 for y. This shows that 1.32 is not the exact value for y. Substituting 1.33 for x gives 0.0226 for y. Put these results in tabular form

	x	y
P	1.32	-0.0200
Q	1.33	$+0.0226$
Differences	0.01	0.0426

This shows that the X-intercept of the graph of the given equation is between the points P and Q (Fig. 86). Thus, a root of $x^3 - x - 1$ is greater than 1.32 and less than 1.33. Now reason as follows: The actual root lies between 1.32 and 1.33, and the zero value of y corresponds to it. This zero is approximately 200/426 of the way between the two values of x. Hence if the curve be nearly straight between $x = 1.32$, and $x = 1.33$, the desired value of x is approximately 200/426 of the way between 1.32 and 1.33 or it is $x = 1.3247$ approximately. This value is probably correct to the fourth decimal place. The next step will show that this result is correct to four decimal places.

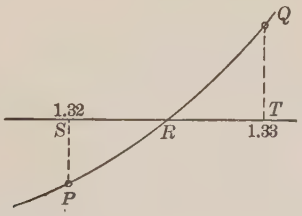


FIG. 86.—Method of approximation to a root of an equation.

To find a third approximation we build another table of values:

	x	y
	1.3247	−0.0000766
	1.3248	+0.0003499
Differences	0.0001	0.0004265

Reasoning as before, we get $x = 1.324718$, which is very likely true to the last decimal place.

This method is applicable to an equation like Exercise 8 above. In fact, it is the only numerical method that is applicable in such cases.

Exercises

Find correct to four decimal places the roots of:

1. $x^3 - 4x + 10 = 0$.

2. $x^3 - 12x - 8 = 0$. (See Exercises 1 and 2, Sec. **96.**)

Miscellaneous Exercises

1. Show that

$$\frac{\tan^2 \alpha}{1 + \tan^2 \alpha} \equiv \sin^2 \alpha.$$

2. Show that

$$\frac{\sqrt{1 - \sin^2 \alpha}}{\sin \alpha} \equiv \frac{\cos \alpha}{\sqrt{1 - \cos^2 \alpha}}.$$

3. Show that $\cot^2 \alpha - \cos^2 \alpha \equiv \cot^2 \alpha \cos^2 \alpha.$

4. Show that

$$\frac{1}{\sqrt{\sec^2 \alpha - 1}} \equiv \sqrt{\csc^2 \alpha - 1.}$$

5. Show that

$$\frac{1 + \tan^2 \alpha}{1 + \cot^2 \alpha} \equiv \frac{\sin^2 \alpha}{\cos^2 \alpha}.$$

6. Show that

$$\sqrt{\frac{1 - \sin \alpha}{1 + \sin \alpha}} \equiv \sec \alpha - \tan \alpha.$$

7. Show that

$$\frac{\sin \alpha}{1 + \cos \alpha} + \frac{1 + \cos \alpha}{\sin \alpha} \equiv 2 \csc \alpha.$$

8. Show that

$$\frac{\csc \alpha}{\cot \alpha + \tan \alpha} \equiv \cos \alpha.$$

9. Show that

$$\frac{1}{\cot u + \tan u} \equiv \sin u \cos u.$$

10. Show that

$$\csc^4 u \,(1 - \cos^4 u) - 2 \cot^2 u \equiv 1.$$

CHAPTER VI

PERMUTATIONS AND COMBINATIONS; THE BINOMIAL THEOREM

98. Fundamental Principle.—*If one thing can be done in n different ways and another can be done in r different ways, then both things can be done together, or in succession, in $n \times r$ different ways.* This simple theorem is fundamental to the work of this chapter. To illustrate, if there be three ways of going from Madison to Chicago and seven ways of going from Chicago to New York, then there are twenty-one ways of going from Madison to New York.

To prove the general theorem, note that if there be *only one* way of doing the first thing, *that* way could be associated with each of the r ways of doing the second thing, making r ways of doing both. That is, for each way of doing the first, there are r ways of doing both things; hence, for n ways of doing the first there are $n \times r$ ways of doing both.

Illustrations: A penny may fall in two ways; a common die may fall in six ways; the two may fall together in twelve ways.

In a society, any one of nine seniors is eligible for president and any one of fourteen juniors is eligible for vice-president. The number of tickets possible is, therefore, 9×14, or 126.

I can purchase a present at any one of four shops. I can give it to any one of seven people. I can, therefore, purchase and give it away in any one of twenty-eight different ways.

A product of two factors is to be made by selecting the first factor from the numbers a, b, c, and then selecting the second factor from the numbers x, y, z, u, v. The number of possible products is, therefore, 15.

If a first thing can be done in n different ways, a second in r different ways, and a third in s different ways, the three things can be done in $n \times r \times s$ different ways. This follows at once

from the fundamental principle, since we may regard the first two things as constituting a single thing that can be done in nr ways, and then associate it with the third, making $nr \times s$ ways of doing the *two* things, consisting of the first two *and* the third.

In the same way, if one thing can be done in n different ways, a second in r different ways, a third in s, a fourth in t, etc., then all can be done together in $n \times r \times s \times t \ . \ . \ .$ different ways.

Thus, n different presents can be given to x men and a women in $(x + a)^n$ different ways, for the first of the n presents can be given away in $x + a$ different ways, the second can be given away in $x + a$ different ways, and the third in $x + a$ different ways and so on. Hence, the number of possible ways of giving away the n presents to $x + a$ men and women is

$$(x + a)(x + a)(x + a) \ . \ . \ . \text{ to } n \text{ factors, or } (x + a)^n.$$

Exercises

1. A building has six exits. In how many ways can a person leave the building and enter by a different door?

2. A car has five seats. In how many different ways may three people be seated, each occupying a different seat?

3. In how many different ways may three presents be given away to ten people?

99. Definitions.—Every distinct order in which objects may be placed in a line or row is called a **permutation,** or an **arrangement.** Every distinct selection of objects that can be made, irrespective of the order in which they are placed, is called a **combination,** or **group.**

Thus, if we take the letters a, b, c, two at a time, there are six arrangements, namely, ab, ac, ba, bc, ca, cb, but there are only three groups, namely, ab, ac, bc.

If we take the three letters *all* at a time, there are six arrangements possible, namely, abc, acb, bca, bac, cab, cba, but there is only *one* group, namely, abc.

Permutations and combinations are both results of mode of selection. Permutations are selections made with the understanding that two selections are considered as different *even though they differ in arrangement only;* combinations are selections made

with the understanding that two selections are not considered as different, *if they differ in arrangement only.*

In the following work, products of the natural numbers like

$$1 \times 2 \times 3;\ 1 \times 2 \times 3 \times 4 \times 5;\ \text{etc.}$$

are of frequent occurrence. These products are abbreviated by the symbols 3!, 5! and read **"factorial three," "factorial five"** respectively.

100. Formula for the Number of Permutations of n Different Things Taken All at a Time.—We are required to find how many possible ways there are of arranging n different things in a line. Lay out a row of n blank spaces, so that each may receive one of these objects, thus:

1		2		3		4		5	. . .	n

In the first space we may place any one of the n objects; therefore, that space may be occupied in n different ways. The second space, after one object has been placed in the first space, may be occupied in $n - 1$ different ways; hence, by the fundamental principle, the two spaces may be occupied in $n(n - 1)$ different ways. In like manner, the third space may be occupied in $n - 2$ different ways, and, by the same principle, the first three spaces may be occupied in $n(n - 1)(n - 2)$ different ways, and so on. The next to the last space can be occupied in but two different ways, since there are but two objects left, and the last space can be occupied in but one way by placing therein the last remaining object. Hence, the total number of different ways of occupying the n spaces in the row with the n objects is the product

$$n(n - 1)(n - 2) \ . \ . \ . \ 3 \cdot 2 \cdot 1,$$

or

$$n!$$

If we use the symbol P_n to stand for the number of permutations of n things taken all at a time, then we write

$$\mathbf{P_n = n!} \qquad (1)$$

101. Formula for the Number of Permutations of n Things Taken r at a Time.—We are required to find how many possible ways there are of arranging a row consisting of r different things,

when we may select the r things from a larger group of n different things.

For convenience in reasoning, lay out a row of r blank spaces, so that each of the spaces may receive one of the objects, thus:

| 1 | | 2 | | 3 | . . . | $r-1$ | | |

In the first space of the row, we may place any one of the n objects; therefore, that space may be occupied in n different ways. The second space, after one object has been placed in the first space, may be occupied in $n - 1$ different ways; hence, by the fundamental principle, the two spaces may be occupied in $n(n - 1)$ different ways. In like manner, the third space may be occupied in $n - 2$ different ways; hence, the first three may be occupied in $n(n - 1)(n - 2)$ different ways, and so on. The last, or rth, space can be occupied in as many different ways as there are objects left. When an object is about to be selected for the rth space, there have been used $r - 1$ objects (one for each of the $r - 1$ spaces already occupied). Since there were n objects to begin with, the number of objects left is $n - (r - 1)$, or $n - r + 1$, which is the number of different ways in which the last space in the row may be occupied. Hence, the formula:

$$\mathbf{P_{n,r} = n(n - 1)(n - 2) \ . \ . \ . \ (n - r + 1),} \qquad (1)$$

in which $P_{n,r}$ stands for the number of permutations of n things taken r at a time.

This formula, by multiplication and division by $(n - r)!$, becomes:

$$P_{n,r} = \frac{n(n-1) \ . \ . \ . \ (n-r+1)(n-r)(n-r-1) \ . \ . \ . \ 3 \cdot 2 \cdot 1}{(n-r)(n-r-1) \ . \ . \ . \ 3 \cdot 2 \cdot 1}$$

or

$$\mathbf{P_{n,r} = \frac{n!}{(n - r)!}.} \qquad (2)$$

This formula is more compact than Eq. (1) above, but the fraction is not in its lowest terms.

Formula (1) is easily remembered by the fact that there are exactly r factors, beginning with n and decreasing by one. Thus we have

$$P_{10,7} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4.$$

Exercises

1. How many permutations can be made of six things taken all at a time?

2. How many different numbers can be made with five digits 1, 2, 3, 4, 5, using each digit once and only once to form each number?

3. The number of permutations of four things taken all at a time bears what ratio to the number of permutations of seven things taken all at a time?

4. How many arrangements can be made of eight things taken three at time?

5. How many arrangements can be made of eight things taken five at a time?

6. How many four-figure numbers can be formed with the nine digits 1, 2, . . . 9 without repeating any digit in any number?

7. How many different signals can be made with seven different flags, by hoisting them one above another five at a time?

8. How many different signals can be made with seven different flags, by hoisting them one above another *any number* at a time?

9. How many different arrangements can be made of nine ball players, supposing only two of them can catch and only one can pitch?

10. How many different ways may the letters of the word *algebra* be written, using all of the letters?

102. Formula for the Number of Combinations, or Groups, of n Different Things Taken r at a Time.—It is obvious that the number of combinations, or groups, consisting of r objects each that can be selected from n objects, is less than the number of permutations of the same objects taken r at a time, for each combination or group when selected can be arranged in a large number of ways. In fact, since there are r objects in the group, each group can be arranged in exactly $r!$ different ways. Hence, for each group of r objects, selected from n objects, there exists $r!$ permutations of r objects each. Therefore, the number of permutations of n things taken r at a time is $r!$ times the number of combinations of n objects taken r at a time. Calling the unknown number of combinations x, we have

$$x \times r! = P_{n,r} = \frac{n!}{(n-r)!},$$

or, solving for x,

$$x = \frac{n!}{r!(n-r)!}.$$

This is the number of combinations of n objects taken r at a time, and may be symbolized

$$C_{n,r} = \frac{n!}{r!(n-r)!}. \tag{1}$$

This fraction will always reduce to a whole number. It may be written in the useful form

$$C_{n,r} = \frac{n(n-1)(n-2) \ . \ . \ . \ (n-r+1)}{1 \times 2 \times 3 \ . \ . \ . \ r}. \tag{2}$$

It is easily remembered in this form, for it has r factors in both the numerator and the denominator. Thus for the number of combinations of ten things taken four at a time we have four factors in the numerator and denominator, or

$$C_{10,4} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4}.$$

Exercises

1. How many different products of three each can be made with the five numbers a, b, c, d, e, provided each combination of three factors gives a different product?

2. How many products can be made from nine different numbers, by taking six numbers to form each product?

3. How many products can be made from nine different numbers, by taking four numbers to form each product?

4. How many different hands of thirteen cards each can be held at a game of whist?

5. A building has five entrances. In how many ways can a person enter the building and leave by a different door?

6. In how many ways can a child be named, supposing that there are 400 different Christian names, without giving it more than three names?

7. In how many ways can a committee of three be appointed from six Italians, four Frenchmen, and seven Americans, provided each nationality is represented?

8. There are five straight lines in a plane, no two of which are parallel; how many intersections are there?

9. There are five points in a plane, no three of which are collinear; how many lines result from joining each point to every other point?

10. In a plane there are n straight lines, no two of which are parallel; how many intersections are there?

11. In a plane there are n points, no three of which are collinear; how many straight lines do they determine?

12. In a plane there are n points, no three of which are collinear, except r, which are all in the same straight line; find the number of straight lines which result from joining them.

13. In how many ways can seven people sit at a round table?

14. In how many ways can seven beads of different colors be strung so as to form a bracelet?

15. How many different sums of money can be formed from a dime, a quarter, a half dollar, a dollar, a quarter eagle, a half eagle, and an eagle?

103. Binomial Expansion.—The demonstration of the binomial theorem may be based upon the following law of multiplication: *The product of any number of polynomials is the aggregate of all the possible partial products which can be made by taking one term and only one from each of the polynomials.* This statement is merely a definition of what is meant by the product of two or more polynomials (see Chap. XV, Sec. **305**). Thus,

$$(x + a)(y + b)(z + c) =$$
$$xyz + ayz + bxz + cxy + abz + bcx + cay + abc.$$

Each of the eight partial products contains a letter from each parenthesis, and never two from the same parenthesis. The number of terms is the number of different ways in which a letter can be selected from each of the three parentheses. In the present case this is, by Sec. **98**, $2 \times 2 \times 2 = 8$.

Let it be required to write out the value of $(x + a)^n$, where x and a stand for any two numbers and n is a positive integer. That is, we must consider the product of the n parentheses

$$(x + a)(x + a)(x + a) \ . \ . \ . \ (x + a),$$

by the distributive law stated above.

First.—Take an x from each of the parentheses to form *one* of the partial products. This gives the term x^n of the product.

Second.—Take an a from the first parenthesis with an x from each of the other $(n - 1)$ parentheses. This gives $ax^{(n-1)}$ as another partial product. But if we take a from the second parenthesis and an x from each of the other $n - 1$ parentheses, we get ax^{n-1} as another partial product. Likewise by taking a from *any* of the

parentheses and an x from each of the other $(n-1)$ parentheses, we shall obtain ax^{n-1} as a partial product. Hence, the final product contains n terms like ax^{n-1}, or, adding these, we obtain nax^{n-1} *as a part of the product.*

Third.—We may obtain a partial product like a^2x^{n-2} by taking an a from *any two* of the parentheses, together with the x's from each of the other $n-2$ parentheses. Hence, there are as many partial products like a^2x^{n-2} as there are ways of selecting two a's from n parentheses; that is, as many ways as there are groups, or combinations, of n things taken two at a time, or

$$\frac{n(n-1)}{1\cdot 2}.$$

Hence, $\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}$ is another part of the product.

Fourth.—We may obtain a partial product like a^3x^{n-3} by taking an a from any three of the parentheses together with the x's from each of the other $n-3$ parentheses. Hence, there are as many partial products like a^3x^{n-3} as there are ways of selecting three a's from n parentheses, that is, as many ways as there are combinations of n things taken three at a time, or $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$. Hence, $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3x^{n-3}$ is another part of the product.

In general, we may obtain a partial product like a^rx^{n-r} (where r is an integer $< n$) by taking an a from *any* r parentheses together with the x's from each of the other $n-r$ parentheses. Hence, there are as many partial products like a^rx^{n-r} as there are ways of selecting r a's from n parentheses; that is, as many ways as there are combinations of n things taken r at a time, or $\frac{n!}{r!(n-r)!}$. Hence, $\frac{n!}{r!(n-r)!}a^rx^{n-r}$ stands for any term in general in the product $(x+a)^n$.

Finally, we may obtain one partial product like a^n by taking an a from each of the parentheses. Hence, a^n is the last term in the product.

Thus we have shown that

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2x^{n-2} + \ldots + \frac{n!}{r!(n-r)!} a^r x^{n-r} + \ldots + a^n. \quad (1)$$

This is the binomial formula of Isaac Newton. The right-hand side is called the expansion or development of the power of the binomial.

It is obvious that the expansion of $(x - a)^n$ will differ from the above only in the signs of the alternate terms containing the odd powers of a, which, of course, will have the negative sign.

Exercises

1. Expand $(u + 3y)^5$. Here $x = u$ and $a = 3y$. By the formula we get

$$u^5 + 5u^4(3y) + 10u^3(3y)^2 + 10u^2(3y)^3 + 5u(3y)^4 + (3y)^5.$$

Performing the indicated operations, we obtain

$$u^5 + 15u^4y + 90u^3y^2 + 270u^2y^3 + 405uy^4 + 243y^5.$$

Expand each of the following by the binomial formula:

2. $(r^2 - 2)^4$.
3. $(3b - \frac{1}{2})^5$.
4. $(c + x)^6$.
5. $(2x^2 - x)^6$.
6. $(1 - a)^8$.
7. $(-x + 2a)^7$.
8. $(x^{3/2} + x^{3/4})^6$.
9. $(a^{-2} - b^{1/2})^4$.
10. $(\sqrt{ab} - \sqrt[3]{ab})^6$.
11. $(\frac{1}{2} + x)^5$.
12. $(b^2 - c^2)^5$.
13. $(3a + \frac{1}{2})^9$.
14. $(5d - 3y)^5$.
15. $(3x^3 - 1)^4$.
16. $(\sqrt{a} + x)^6$.
17. $(a + [x + y])^3$
18. $(a + b - y)^3$.
19. $(x^2 + 2ax + a^2)^3$.

104. Binomial Expansion for Fractional and Negative Exponents.—It is proved in calculus that

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!} x^2 \pm \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$$

is true for fractional and for negative values of n, provided x be less than 1 in absolute value. The number of terms in the expansion is not finite, but is *unlimited*.

By the above formula, we have

$$\sqrt{1+x} = 1 + \binom{1}{2}x + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}x^2 + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$
$$= 1 + \binom{1}{2}x - \binom{1}{8}x^2 + \binom{1}{16}x^3 - \binom{5}{128}x^4.$$

If

$$x = \tfrac{1}{2}$$

this becomes

$$\sqrt{\tfrac{3}{2}} = 1 + \tfrac{1}{4} - \tfrac{1}{32} + \tfrac{1}{128} - \tfrac{5}{2048} + \cdots$$

Therefore, using five terms of the expression

$$\sqrt{\tfrac{3}{2}} = \tfrac{25097}{20048} = 1.2241 \text{ approximately.}$$

The square root, correct to five figures, is really 1.2247. Thus, the error in this case is less than $\frac{1}{10}$ of per cent if only five terms of the series be used. The degree of accuracy in each case is dependent both upon the value of n and upon the value of x. Obviously, for a given value of n, the series converges for small values of x more rapidly than for larger values.

As another example, suppose it be required to expand $(1 - x)^{-1}$. By the binomial theorem

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-1-1)}{2!}(-x)^2$$
$$+ \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 + \cdots$$
$$= 1 + x + x^2 + x^3 + \cdots$$

If five terms of the series be used, the error is $\frac{1}{16}$ for $x = \frac{1}{2}$, or about 6 per cent.

105. Approximation Formulas.—If x be very small, the expansion of

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

is approximately

$$(1+x)^n \doteqdot 1 + nx, \tag{1}$$

since x^2, x^3, and all higher powers of x are much smaller than x. Thus using the symbol $\doteqdot$ to express "approximately equals," we have, for example,

$$(1.01)^3 \doteqdot 1.03,$$

for,

$$(1 + \tfrac{1}{100})^3 \doteqdot 1 + \tfrac{3}{100}.$$

The true value of $(1.01)^3$ is 1.030301, so that the approximation is very good.

Likewise,

$$(1 - x)^n \doteq 1 - nx, \tag{2}$$

if x be small.

If x, y, and z be small compared with unity, the following approximation formulas hold:

$$(1 + x)(1 + y) \doteq 1 + x + y, \tag{3}$$

$$\frac{1 + x}{1 + y} \doteq 1 + x - y, \tag{4}$$

$$(1 + x)(1 + y)(1 + z) \doteq 1 + x + y + z. \tag{5}$$

The approximation formulas are proved as follows:

$(1 + x)(1 + y) = 1 + x + y + xy \doteq 1 + x + y$, for xy is small compared to x and y.

$\dfrac{(1 + x)}{(1 + y)} = 1 + x - y + \dfrac{y^2 - xy}{1 + y} \doteq 1 + x - y$, for the fraction is small compared to x and y.

$(1 + x)(1 + y)(1 + z) \doteq (1 + x + y)(1 + z) \doteq 1 + x + y + z.$

Exercises

1. Explain the following approximation formulas, in which $|x| < 1$

$$\sqrt{1 + x} \doteq 1 + \tfrac{1}{2}x.$$
$$\sqrt{1 - x} \doteq 1 - \tfrac{1}{2}x.$$
$$(1 + x)^{-\frac{1}{2}} \doteq 1 - \tfrac{1}{2}x.$$
$$\sqrt[3]{1 + x} \doteq 1 + \tfrac{1}{3}x.$$
$$\sqrt[3]{1 - x} \doteq 1 - \tfrac{1}{3}x.$$
$$(1 + x)^{-\frac{2}{3}} \doteq 1 - \tfrac{2}{3}x.$$
$$(1 + x^2)^{\frac{1}{3}} \doteq 1 + \tfrac{1}{3}x^2.$$

2. Compute the approximate numerical value of the following:

(*a*) $(1.03)^{\frac{1}{2}}$.

(*b*) $(1.02)(1.03)$.

(*c*) $\dfrac{(1.01)(1.02)}{(1.03)(1.04)}$.

(*d*) $(1.05)^{\frac{1}{3}}$.

(*e*) $\dfrac{1.02}{1.03}$.

3. The formula for the period of a simple pendulum is

$$T = \pi\sqrt{\frac{l}{g}}.$$

For the value of gravity at New York, this reduces to

$$T = \frac{\sqrt{l}}{6.253},$$

in which l, the length of the pendulum, is measured in inches. This pendulum beats seconds when

$$l = (6.253)^2, \text{ or } 39.10 \text{ in.}$$

What is the period of the pendulum if l be lengthened to 39.13 in.?

Hint:

$$T = \frac{\sqrt{l}}{6.253}$$

$$T' = \frac{\sqrt{l+h}}{6.253} = \frac{\sqrt{l}}{6.253}\sqrt{1+\frac{h}{l}}$$

$$= \frac{\sqrt{l}}{6.253}\left(1+\frac{h}{2l}\right).$$

Take $l = 39.10$, and $h = 0.03$.

Then

$$T' = 1 + \frac{0.03}{78.20}$$

$$= 1.00038.$$

A day contains 86,400 sec. The change of length would, therefore, cause a loss of 32.8 sec. per day, if the pendulum were attached to a clock.

4. On the ocean how far can one see at an elevation of h ft. above its surface?

Call the radius of the earth a(= 3960 miles), and the distance one can see d, which is along a tangent from the point of observation to the sphere. Since h is in feet, and since $a + h/5280$, d, and a are the sides of a right triangle, we have $(a + h/5280)^2 = d^2 + a^2$,

or

$$a^2\left[1 + \frac{h}{5280a}\right]^2 = d^2 + a^2.$$

Expanding the binomial by the approximation formula we have

$$a^2\left[1 + \frac{2h}{5280a}\right] = d^2 + a^2,$$

or

$$\begin{aligned} d^2 &= \frac{2ah}{5280} \\ &= 2 \times \frac{3960h}{5280} \\ &= \tfrac{3}{2}h, \end{aligned}$$

or

$$d = \sqrt{\tfrac{3}{2}h},$$

where d is expressed in miles and h in feet.

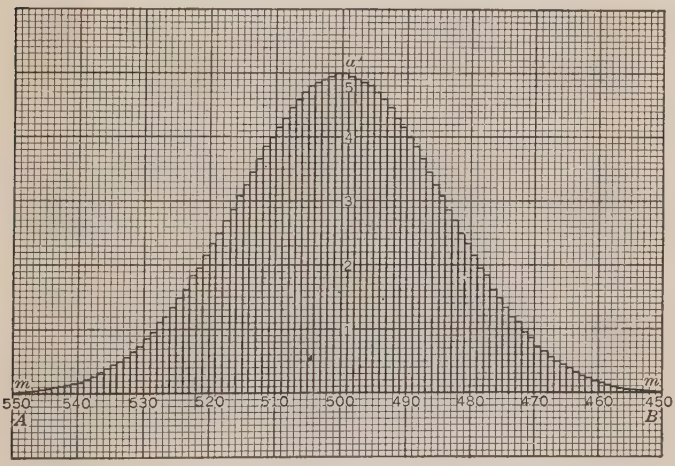


FIG. 87.—Graphical representation of the values of the binomial coefficients in the 999th power of a binomial. The middle coefficients are taken equal to 5, for convenience, and the others are expressed to that scale also.

5. By what per cent is the area of a circle altered if its radius of 100 cm. be changed to 101 cm.?

6. By what per cent is the volume of a sphere, $\frac{4}{3}\pi a^3$, altered if the radius be changed from 100 to 101 cm.?

7. If the formula for the horsepower of a ship is i.hp. $= S^3D^{\frac{2}{3}}/200$, where S is speed in knots and D is displacements in tons, what increase

in horsepower is required in order to increase the speed from 15 to 16 knots, the tonnage remaining constant at 5000? What increase in horsepower is required to maintain the same speed if the load or tonnage be increased from 5000 to 5500?

106.* Graphical Representation of the Coefficients of Any Power of a Binomial.—If we erect ordinates at equal intervals on the X-axis proportional to the coefficients of any power of a binomial, we find that a curve is approximated, which becomes very striking as the exponent is taken larger and larger. In Fig. 87 the ordinates are proportional to the coefficients of the 999th power of $x + a$. The drawing is due to Quetelet.

The limit of the broken line at the top of the ordinates in Fig. 87 is, as n is increased indefinitely, a bell-shaped curve, known as the **probability curve.** In treatises on the Theory of Probability, it is shown that the equation of the curve is $y = ae^{-bx^2}$. Note that Fig. 87 shows only a small central portion of the whole graph, which extends from $m = 0$ to $m = 1000$. The ordinates from $m = 0$ to $m = 450$ and from $m = 550$ to $m = 1000$ are negligibly small compared to the ordinates shown in Fig. 87.

CHAPTER VII

PROGRESSIONS

107. An Arithmetical Progression, or an Arithmetical Series, is any succession of terms such that each term differs from that immediately preceding by a fixed number called the **common difference.** The following are arithmetical progressions:

$$1, 2, 3, 4, 5.$$
$$4, 6, 8, 10, 12.$$
$$32, 27, 22, 17, 12.$$
$$2\tfrac{1}{2}, 3\tfrac{3}{4}, 5, 6\tfrac{1}{4}, 7\tfrac{1}{2}.$$
$$(u - v), u, (u + v).$$
$$a, a + d, a + 2d, a + 3d, \ldots$$

The first and last terms are called the **extremes,** and the other terms are called the **means.**

Where there are but three numbers in the series, the middle number is called the **arithmetical mean** of the other two. To find the arithmetical mean of the two numbers a and b, proceed as follows:

Let A stand for the required mean; then, by definition,

$$A - a = b - A,$$

whence

$$A = \frac{a + b}{2}.$$

Thus, the arithmetical mean of 12 and 18 is 15, for 12, 15, 18 is an arithmetical progression of common difference 3.

By the **arithmetical mean,** or **arithmetical average,** of several numbers is meant the result of dividing the sum of the numbers by the number of the numbers. It is, therefore, such a number that if all numbers of the set were equal to the arithmetical mean, the sum of the set would be the same.

The n terms,

$$a,\ (a + d),\ (a + 2d),\ (a + 3d),\ \ldots\ (a + [n - 1]d)$$

constitute a general arithmetical progression. Note that any term, such as the *fifth*, is obtained by adding *four* times the common difference to the initial term.

Here a and d may be any algebraic numbers whatsoever, integral or fractional, rational or irrational, positive or negative, but n must be a positive integer. When the common difference is negative, the progression is said to be a **decreasing progression**; otherwise, it is an **increasing progression.**

From the general progression written above, we see that a formula for the nth term of any arithmetical progression may be written

$$\mathbf{l = a + (n - 1)d,} \qquad (1)$$

in which l stands for the nth term.

Equation (1) enables us to obtain the value of any one of the numbers, l, a, n, d, when the other three are given. Thus:

1. Find the one-hundredth term of

$$3 + 8 + 13 + \ldots$$

Here

$$a = 3,\ d = 5,\ n = 100.$$

Therefore,

$$l = 3 + 99 \times 5 = 498.$$

2. Find the number of terms in the progression

$$5 + 7 + 9 + \ldots + 39.$$

Here

$$a = 5,\ d = 2,\ l = 39.$$

Therefore,

$$39 = 5 + (n - 1)2,$$

or

$$n = 18.$$

3. Find the common difference in a progression of fifteen terms in which the extremes are $\frac{1}{2}$ and $42\frac{1}{2}$.

Here

$$a = \tfrac{1}{2},\ l = 42\tfrac{1}{2},\ n = 15,$$

whence,

$$42\tfrac{1}{2} = \tfrac{1}{2} + (15 - 1)d,$$

or

$$d = 3.$$

108. The Sum of *n* Terms.—If s stands for the sum of n terms of an arithmetical progression, and if the sum of the terms be written first in natural order, and again in reverse order, we have

$$s = a + (a + d) + (a + 2d) + \ldots + (a + [n - 1]d), \quad (1)$$
$$s = l + (l - d) + (l - 2d) + \ldots + (l - [n - 1]d). \quad (2)$$

Adding Eqs. (1) and (2), term by term, and noting that the positive and the negative common differences nullify one another, we obtain

$$2s = (a + l) + (a + l) + (a + l) + \ldots + (a + l), \quad (3)$$

or, since the number of terms in the original progression is n, we may write

$$2s = n(a + l),$$

or

$$\mathbf{s = \frac{n(a + l)}{2}}. \quad (4)$$

If the value for l, from Eq. (1) Sec. **107,** be substituted in Eq. (4) it becomes

$$\mathbf{s = \frac{n}{2}[2a + (n - 1)d]}. \quad (5)$$

In Eq. (4), $(a + l)/2$ is the average of the first and the nth terms. The formula (4) states, therefore, that the sum equals the number of the terms multiplied by the average of the first and last.

An arithmetical progression is a very simple particular instance of a much more general class of expressions known in mathematics as **series.** A series is any sequence of terms formed according to some law, such as:

$$(x + 1) + (x + 2)^2 + (x + 3)^3 + \ldots$$
$$x + 3x^3 + 5x^5 + \ldots$$
$$\cos x + \cos 2x + \cos 3x + \ldots$$

It is only in a very limited number of cases that a short expression can be found for the sum of n terms of a series. An arithmetical progression is one of these cases.

Formula (4) enables us to find the value of any one of the numbers s, n, a, l, when the values of the other three are given. Thus:

1. Find the number of terms in an arithmetical progression in which the first term is 4, the last term 22, and the sum 91.

Here

$$a = 4,\ l = 22,\ s = 91,$$

whence,

$$91 = \frac{n(4 + 22)}{2},$$

or

$$n = 7.$$

The two formulas, Eq. (1), Sec. **107,** and Eq. (4), Sec. **108,** contain five letters; hence, if any two of them stand for unknown numbers, and the values of the others are given, the values of the two unknown numbers can be found by the solution of a system of two equations. Thus:

2. Find the number of terms in the progression whose sum is 1095, if the first term be 38 and the difference be 5.

Here

$$s = 1095,\ a = 38, \text{ and } d = 5.$$

whence,

$$l = 38 + (n - 1)5, \tag{6}$$

$$1095 = \frac{n(38 + l)}{2}. \tag{7}$$

From Eq. (6)

$$l = 33 + 5n. \tag{8}$$

From Eq. (7)

$$2190 = 38n + nl. \tag{9}$$

Substituting the value of l from Eq. (8) in Eq. (9), we get

$$2190 = 71n + 5n^2. \tag{10}$$

Solving this quadratic equation, we find

$$n = 15, \text{ or } -29.2.$$

The second result is inadmissible, since the number of terms cannot be either negative or fractional.

Exercises

Solve each of the following:

1. Given, $a = 7,\ d = 4,\ n = 15$; find l and s.

2. Given, $a = 17,\ l = 350,\ d = 9$; find n and s.

3. Given, $a = 3,\ n = 50,\ s = 3825$; find l and d.

4. Given, $s = 4784$, $a = 41$, $d = 2$; find l and n.

5. Given, $s = 1008$, $d = 4$, $l = 88$; find a and n.

6. Find the sum of the first n even numbers.

7. Find the sum of the first n odd numbers.

8. Insert nine arithmetical means between $-\frac{7}{8}$ and $+\frac{7}{8}$.

9. Sum $(a + b)^2 + (a^2 + b^2) + (a - b)^2$ to n terms.

10. Find the sum of the first fifty multiples of 7.

11. Find the amount of \$1 at simple interest at 5 per cent for 1920 yrs.

12. How long must \$1 accumulate at $3\frac{1}{2}$ per cent simple interest until the total amounts to \$100?

13. How many terms of the progression $9 + 13 + 17 + \ldots$ must be taken in order that the sum may equal 624? How many terms must be taken in order that the sum may exceed 750?

14. Show that the only right triangles whose sides are in arithmetical progression are those whose sides are proportional to 3, 4, and 5.

109. A geometrical progression or a geometrical series is any succession of terms such that each term is the product of the preceding term by a fixed factor called the **ratio.** The following are examples:

$3, 6, 12, 24, 48.$

$100, -50, 25, -12\frac{1}{2}.$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}.$

$a, ar, ar^2, ar^3, ar^4 \ldots$

The **geometrical mean** G of two numbers, a and b, is a number such that a, G, b is a geometrical progression. By definition

$$\frac{G}{a} = \frac{b}{G},$$

whence,

$$G^2 = ab,$$

or

$$G = \sqrt{ab}.$$

Thus, 4 is the geometrical mean of 2 and 8. The *arithmetical mean* of 2 and 8 is 5. The **geometrical mean of n positive numbers** is the value of the nth root of their product. Thus, the geometrical mean of 8, 9 and 24 is $\sqrt[3]{8 \times 9 \times 24} = 12$.

110. The *n*th Term and the Sum of *n* Terms.—If a represents the first term and r the ratio of any geometrical progression, the progression consists of the n terms:

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ ar^{n-2},\ ar^{n-1}.$$

Note that any term, such as the *fifth*, is obtained by multiplying the first term a by the *fourth* power of the ratio r. Therefore, representing the nth term by l, we obtain the simple formula

$$\mathbf{l = ar^{n-1}}. \tag{1}$$

Representing by s the sum of n terms of any geometrical progression, we have

$$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1},$$

or

$$s = a(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}).$$

But, by a fundamental theorem in factoring,[1] the expression in the parenthesis is the quotient of $1 - r^n$ by $1 - r$. Hence,

$$\mathbf{s = \frac{a(1 - r^n)}{(1 - r)}}. \tag{2}$$

Another form is obtained by introducing l by the substitution

$$ar^{n-1} = l,$$

which gives

$$\mathbf{s = \frac{a - rl}{1 - r}}. \tag{3}$$

Formula (1) or (2) enables one to find any one of the four numbers involved in the equations when three are given. The two formulas (1) and (2) considered as simultaneous equations enable one to find any two of the five numbers a, r, n, l, s, when the other three are given. But if r be one of the unknown numbers, the equations of the system may be of a high degree and beyond the range of Chap. V unless solved by graphic means. If n be an unknown number, an equation of a new type is introduced, namely, one with the unknown number appearing as an exponent. Equations of this type, known as exponential equations, will be treated in the chapter on logarithms. The following examples illustrate cases in which the resulting single and simultaneous equations are readily solved.

[1] See Appendix, Chap. XV, Sec. **306.**

1. Insert three geometrical means between 31 and 496.

Here

$$a = 31,\ l = 496, \text{ and } n = 5.$$

Hence,

$$496 = 31 \times r^4$$
$$r^4 = 16,$$

or

$$r = \pm 2.$$

Consequently, the required means are either 62, 124, and 248, or -62, $+124$, and -248.

2. Find the sum of a geometrical progression of five terms, the extremes being 8 and 10,368.

Here

$$a = 8,\ l = 10{,}368, \text{ and } n = 5.$$

Hence,

$$10{,}368 = 8r^4 \tag{1}$$

and

$$s = \frac{10{,}368r - 8}{r - 1}. \tag{2}$$

From the first,

$$r = 6,$$

whence, from the second,

$$s = 12{,}440.$$

3. Find the extremes of a geometrical progression whose sum is 635, if the ratio be 2 and the number of terms be 7.

Here

$$s = 635,\ r = 2, \text{ and } n = 7.$$

Hence,

$$l = a2^6, \tag{3}$$
$$635 = 2l - a. \tag{4}$$

Substituting l from Eq. (3) in Eq. (4), we get

$$635 = 128a - a.$$

Hence,

$$a = 5, \text{ and } l = 320.$$

4. The fourth term of a geometrical progression is 4, and the sixth term is 1. What is the tenth term?

Here

$$ar^3 = 4, \tag{5}$$

and

$$ar^5 = 1. \tag{6}$$

Dividing Eq. (6) by Eq. (5) we obtain

$$r^2 = \tfrac{1}{4}, \text{ or } r = \pm\tfrac{1}{2}.$$

Therefore, from Eq. (5),

$$a = \frac{4}{r^3} = \pm 32.$$

Then the tenth term is

$$\pm 32(\pm\tfrac{1}{2})^9 = \tfrac{1}{16}.$$

Exercises

1. Find the sum of seven terms of $4 + 8 + 16 + \ldots$

2. Find the sum of $-4 + 8 - 16 + \ldots$ to six terms.

3. Find the tenth term and the sum of ten terms of $4 - 2 + 1 - \ldots$

4. Find r and s; given $a = 2$, $l = 31{,}250$, $n = 7$.

5. Insert two geometrical means between 47 and 1269.

6. Insert three geometrical means between 2 and 3.

7. Insert seven geometrical means between a^8 and b^8.

8. Show that the quotient $(a^n - b^n)/(a - b)$ is a geometrical progression.

9. Sum $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots$ to n terms.

10. Sum $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots$ to n terms.

11. Sum $a + ar^{-1} + ar^{-2} + \ldots$ to n terms.

12. If $a, b, c, d, \ldots$ are in geometrical progression, show that $a^2 + b^2, b^2 + c^2, c^2 + d^2, \ldots$ are also in geometrical progression.

13. If any numbers are in geometrical progression, show that their differences are also in geometrical progression.

14. A man agreed to pay for the shoeing of his horse as follows: 1 cent for the first nail, 2 cents for the second nail, 4 cents for the third nail, and so on until the eight nails in each shoe were paid for. What did the last nail cost? How much did he agree to pay in all?

111. Compound Interest.—Just as the amount of principal and interest of a sum of money at simple interest for n years is expressed by the $(n + 1)$st term of an arithmetical progression, so, in a similar way, the amount of any sum at compound interest for n years is represented by the $(n + 1)$st term of a geometrical progression. Thus, the amount of \$1 at compound interest at 4 per cent for 20 years is given by the expression

$$1(1.04)^{20}.$$

The amount of p dollars for n years at r per cent is

$$p\left(1 + \frac{r}{100}\right)^n.$$

The present value of \$1 due 20 years hence, estimating compound interest at 4 per cent, is

$$\frac{1}{(1.04)^{20}}.$$

The value of \$1, paid annually at the beginning of each year into a fund accumulating at 4 per cent compound interest, is, at the end of 20 years

$$(1.04)^1 + (1.04)^2 + \ . \ . \ . \ + (1.04)^{20},$$

which is the sum of the terms of a geometrical progression of twenty terms.

Problems of this character in compound interest, in compound discount, and in the more complicated problems that proceed therefrom, are basal to the theory of annuities, life insurance, and depreciation of machinery and structures. The computation of the high powers involved necessitates the postponement of such problems until the subject of logarithms has been explained.

112. Infinite Geometrical Progressions.—If the ratio of a geometrical progression be a proper fraction, the progression is said to be a **decreasing progression.** Thus,

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \text{ and } \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}, \tfrac{1}{81}$$

are decreasing progressions. If we increase the number of terms in the first of these progressions the sums will always be less than 2; but the difference $(2 - s)$ will become and remain less than any preassigned number.

Definition.—A constant, a, is called the **limit** of a variable, t, if, as t runs through a sequence of numbers, the difference $(a - t)$ becomes, and remains, numerically smaller than any preassigned number.

By definition, 2 is, therefore, the limit of the first of the above progressions. The sum of n terms of this particular progression should be written down by the student for a number of successive values for n, thus:

Number of terms:

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \ldots \quad 10.$$

Sum: $1,\ 1+\frac{1}{2},\ 1+\frac{3}{4},\ 1+\frac{7}{8},\ 1+\frac{15}{16},\ \ldots\ 1+\frac{511}{512}.$

The sum of n terms differs from 2 by only $(\frac{1}{2})^{n-1}$.

It is easy to show that the sum of every decreasing geometrical progression approaches a fixed limit as the number of terms becomes infinite. Write the formula

$$s = \frac{a - ar^n}{1 - r}$$

in the form

$$s = \frac{a}{1-r} - \frac{ar^n}{1-r}. \tag{1}$$

If we suppose that r is a proper fraction and that n increases without limit, then r^n can be made less than any assigned number; for the value of any power of a proper fraction decreases as the exponent of the power increases. As the other parts of the second fraction in Eq. (1) do not change in value as n changes, the fraction as a whole can be made smaller than any number that can be assigned. Hence, we write

$$\lim_{n \doteq \infty} [s] = \frac{a}{1-r}. \tag{2}$$

The left-hand side is read: "The limit of s as n becomes infinite." The symbol $\doteq$ means: "approaches" or "becomes."

Exercises

As $n \doteq \infty$, find the limit of each of the following:

1. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

Here

$$a = \tfrac{1}{2},\ r = -\tfrac{1}{2},$$

whence, limit $s = \dfrac{\frac{1}{2}}{1 - (-\frac{1}{2})} = \dfrac{1}{3}.$

2. 0.3333 . . .

Here

$$a = \tfrac{3}{10},\ r = \tfrac{1}{10},$$

whence, limit $s = \dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \dfrac{1}{3}.$

3. $9 - 6 + 4 - \ldots$

4. 0.272727 . . .

5. 0.279279279 . . .

6. $\frac{1}{3} - \frac{1}{6} + \frac{1}{12} - \ldots$

7. $4 + 0.8 + 0.16 + \ldots$

8. Express the number 8 as the sum of an infinite geometrical progression whose second term is 2.

113.* Graphic Representation.—To represent the geometrical series $1 + r + r^2 \ldots + r^{n-1}$ graphically, lay off $OM = 1$ on OY, $OS_1 = 1$ on OX, $S_1P_1 = r$ on the unit line, and draw MP_1. Draw the arc P_1S_2 and erect P_2S_2. Draw the arc P_2S_2 and erect P_3S_3. Continue this construction until the perpendicular P_nS_n is erected. The series of trapezoids OMP_1S_1, $S_1P_1P_2S_2$, $S_2P_2P_3S_3$, $\ldots$, $S_{n-1}P_{n-1}P_nS_n$ are similar and, since $P_1S_1 = r \times OM$, it follows that $P_2S_2 = rP_1S_1$, $P_3S_3 = rP_2S_2$, $\ldots$, $P_nS_n = rP_{n-1}S_{n-1}$. Hence we have:

$OM = OS_1 = 1$

$P_1S_1 = S_1S_2 = r \therefore OS_2 = 1 + r =$ sum of 2 terms.

$P_2S_2 = S_2S_3 = r^2 \therefore OS_3 = 1 + r + r^2 =$ sum of 3 terms.

$P_3S_3 = S_3S_4 = r^3 \therefore OS_4 = 1 + r + r^2 + r^3 =$ sum of 4 terms.

. . . . . .

$P_{n-1}S_{n-1} = S_{n-1}S_n = r^{n-1} \therefore OS_n = 1 + r + r^2 + \ldots r^{n-1} =$ sum of n terms.

Figure 88 shows the series whose ratio is $r = 1.2$. Figure 89 shows the series whose ratio is 0.8.

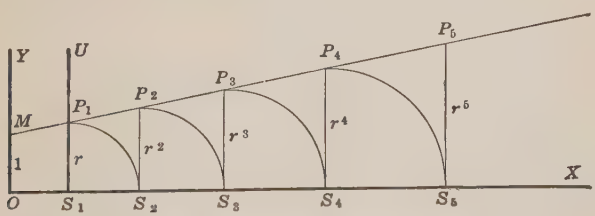


FIG. 88.—Graphical construction of the sum of a *G. P.* $r > 1$.

The line MP_1 has the slope $r - 1$ in Fig. 88 and the slope $-(1 - r)$ in Fig. 89. In each case the Y-intercept is 1. Its equation is, in both cases, $y = (r - 1)x + 1$, or $x = \frac{1 - y}{1 - r}$. In both figures, when $y = P_nS_n = r^n$, $x = OS_n$. Substituting

these values for x and y, we get for the sum of n terms, $s = \frac{1 - r^n}{1 - r}$. Figure 88 shows that, when the number of terms is allowed to increase without limit, the sum OS_n also increases without limit. Figure 89 shows that, when the number of terms is made to increase without limit, the sum OS_n approaches OL as a limit. Now the value of OL is the value of x when $y = 0$. Hence the limit of the sum of the progression, or OL, is $\frac{1}{1 - r}$.

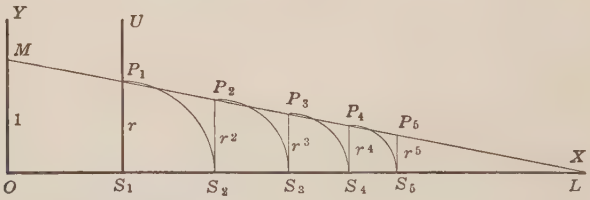


FIG. 89.—Graphical construction of the sum of a $G.\ P.\ r < 1$.

In a geometrical progression whose first term is a, rather than 1 as has been assumed above, the number a occurs as a single constant multiplier in each term and in the expression for s. The general case, therefore, differs in no essential particular from the special case here considered, it being only necessary to enlarge the figures in the ratio a to 1.

Questions and Exercises for Review of Chapters I to VII

1. Define scale; uniform scale; non-uniform scale; arithmetical scale; algebraic scale; double scale.

2. Define constant; variable.

3. Define function; increasing function; decreasing function; even function; odd function.

4. Give illustrations of even functions; of odd functions.

5. Express the area A of an equilateral triangle as a function of the length x of its sides.

6. Express the volume V of a right circular cone as a function of its altitude h. The radius of the base is 10 in.

7. A strip of tin L ft. long and 40 in. wide is made into a gutter with rectangular cross-section, by bending up an equal portion of each side. Express the cross-section y of the gutter as a function of the

breadth x of the amount of tin turned up. Show that the maximum cross-section is 200 sq. in.

8. A strip of tin 24 in. square has an equal square cut from each corner. The rectangular projections are then turned up to form a tray with square base and rectangular sides. If x be the side of the square cut out, show that $4x(12 - x)^2$ is the function representing the volume of the tray.

9. In a triangle whose sides are 6, 8, and 10 ft. is inscribed a rectangle the base of which lies in the longest side of the triangle. Express the area A of the rectangle as a function of its altitude h.

10. A ladder 20 ft. long leans against the vertical wall of a house. Express the area A of the triangle formed by the ladder, the wall, and the horizontal ground, as a function of the distance x of the foot of the ladder from the wall.

11. A right triangle ABC has leg $AC = x$, and leg $BC = \frac{1}{2}x$. Express (a) the area of the triangle as a function of x; (b) the hypotenuse AB as a function of x.

12. Describe the method of representing the position of points on a plane by the rectangular, or Cartesian, system of coordinates. Define axes; origin; abscissa; ordinate; quadrant. How are the quadrants numbered?

13. What is meant by the graph, locus, or curve of an equation?

14. What is meant by the equation of a curve, graph, or locus of a point?

15. Which of the following points are on the curve $3x + 2y = 4$: (a) $(2, -1)$; (b) $(3, 1)$; (c) $(-4, 8)$; (d) $(0, 0)$?

16. Find the distance of each of the following points from the origin: (a) $(1, 3)$; (b) $(-2, 3)$; (c) $(2, -3)$; (d) $(-3, -2)$.

17. Show that, for all values of m, $y = mx$ is a straight line passing through the origin.

18. Show that the equation of any straight line passing through the origin is of the form $y = mx$.

19. Find the equation of a straight line passing through the origin and the point $(-3, 5)$.

20. Show that, for all values of m and b, $y = mx + b$ is the equation of a straight line.

21. Show that the equation of any straight line is of the form $y = mx + b$.

22. Find the equation of a straight line passing through the points $(1, 3)$ and $(-2, 5)$.

23. Define slope of a straight line.

24. Define the Y-intercept and the X-intercept of a straight line.

25. Find the slope, the X-intercept, and the Y-intercept for the following:

(a) $3x + 2y = 6$; (b) $x - 2y = 5$; (c) $2y - 3x = 7$.

26. Define X- and Y-intercepts of a curve.

27. Write the equations of a line if:

(a) Y-intercept is 3 and slope is 2.
(b) Y-intercept is 1 and slope is -2.
(c) Y-intercept is -2 and slope is $\frac{1}{2}$.
(d) X-intercept is 3 and slope is 2.
(e) X-intercept is -2 and slope is 3.
(f) X-intercept is $\frac{1}{2}$ and slope is $-\frac{1}{3}$.
(g) X-intercept is 2 and Y-intercept is 3.

28. What is meant by curves of the parabolic type?

29. What is meant by curves of the hyperbolic type?

30. What is the parabola?

31. What is the equilateral, or rectangular, hyperbola?

32. What is the cubical parabola?

33. What is the semi-cubical parabola?

34. When is a curve symmetrical with respect to the X-axis; with respect to the Y-axis; with respect to the line $x = y$; with respect to the line $y = -x$; with respect to the origin? Give equation of two curves for each of the cases considered above.

35. Sketch, $y = x^2$; $y = \frac{1}{2}x^2$; $y = 2x^2$.

36. Sketch $y^2 = x$; $y^2 = \frac{1}{2}x$; $y^2 = 2x$.

37. Sketch $y = x^2$; $y = -x^2$.

38. Sketch $y = 1/x$; $y = -2/x$; $y = 1/2x$.

39. Sketch $x^2 = y^3$; $x^3 = y^2$.

40. Sketch $y = x^3$; $y = -x^3$.

41. Define rational equation, empirical equation.

42. Write the equation of the curve $y = x^2 - 3x$, after it is translated

(a) 2 units to the right; (b) 3 units to the left; (c) 1 unit up; (d) 5 units down; (e) 1 unit to the left and 2 units down.

43. Find the coordinates of the vertex of:

(a) $y = x^2 + 2x$.
(b) $y = x^2 - 2x + 3$.
(c) $y = 3x^2 + 6x$.
(d) $y = 6x - 3x^2 + 2$.

44. Show that $y = \dfrac{x + 3}{x - 2}$ is an hyperbola. Write the equation of its asymptotes.

45. What is meant by shearing motion?

46. Show that shearing the curve $y = ax^2$ in the line $y = mx$, is equivalent to translating the original curve. Find the coordinates of the vertex of the translated curve.

47. What is meant by the roots of a function?

48. Find the roots of:

$$(a)\ x^2 + 2x - 3;\ (b)\ x^2 - 3x;\ (c)\ 3x^2 + 2x - 6.$$

49. Write the equation of the curve $y^2 = x^3 - x^2$ when reflected in: (*a*) the X-axis; (*b*) the Y-axis; (*c*) the line $x = y$; (*d*) the line $x = -y$.

50. The roots of a function correspond to what points on the graph of the function?

51. Given the functional relationship $2x^2 - 4y^2 = 7$, write the equation which expresses x as an explicit function of y; y as an explicit function of x.

52. Write the equation of a circle, radius a, center at the origin; center at the point (h, k).

53. Show that $x^2 + y^2 + 2gx + 2fy + d = 0$ is a circle.

54. Find the coordinates of the center and the length of the radius of:

(*a*) $x^2 + y^2 - 2x - 4y + 1 = 0$.
(*b*) $x^2 + y^2 + 2x + 4y + 1 = 0$.
(*c*) $x^2 + y^2 + 3x - 4y = 0$.
(*d*) $2x^2 + 2y^2 + 3x + 5y = 0$.
(*e*) $3x^2 + 3y^2 - 6x - \sqrt{2}y = 10$.
(*f*) $x^2 + y^2 + 7x - \sqrt{3}y = 25$.

55. Which circles of Exercise 54 pass through the origin?

56. Write the equation of the circle if:

(*a*) The radius be 5 and the center be at (1, 2).
(*b*) The radius be 6 and the center be at $(-\frac{1}{2}, 2)$.
(*c*) The radius be 10 and the center be at $(-2, -3)$.
(*d*) It pass through the origin and the center be at (1, 1).
(*e*) It pass through the origin and the center be at $(-2, 3)$.
(*f*) It pass through (1, 2) and the center be at $(-2, 3)$.

57. Write the equation of a line passing through the origin and the center of the circle

$$x^2 + y^2 - 2x + 3y = 5.$$

58. Write the equation of a line passing through the point (2, 3) and through the center of the circle

$$x^2 + y^2 - 3x - 2y = 0.$$

59. Show that if two straight lines be mutually perpendicular, the slope of one is the negative reciprocal of the slope of the other.

60. Show that $(x - a)^2 + y^2 = a^2$ and $(x - 3a)^2 + y^2 = a^2$ are tangent to each other.

61. Find analytically the coordinates of the points of intersection of $x^2 + y^2 - 4x - 9y = 9$ and $y = \frac{1}{2}x + 1$.

62 Find approximate solutions for Exercise 61 by drawing the curves on squared paper.

63. Solve graphically the simultaneous equations

$$x^2 + y^2 - 2x - 4y = 4.$$
$$x^2 + y^2 + 4x - 4y = 0.$$

64. Define degree; radian.

65. Define the six circular functions.

66. Express the following as radians:

(*a*) 45°; (*b*) 90°; (*c*) 180°; (*d*) 135°; (*e*) 225°; (*f*) 60°; (*g*) 30°; (*h*) 300°; (*i*) 270°; (*j*) 315°; (*k*) 120°; (*l*) 160°; (*m*) 216°; (*n*) 310°.

67. Express the following radians as degrees:

(*a*) $\frac{1}{4}\pi$; (*b*) $\frac{1}{2}\pi$; (*c*) $\frac{1}{3}\pi$; (*d*) $\frac{3}{2}\pi$; (*e*) $\frac{5}{3}\pi$; (*f*) 3; (*g*) 2.

68. How many revolutions per minute are 10 radians per second? $\frac{1}{2}\pi$ radians per second? $\frac{5}{3}\pi$ radians per second?

69. A car is running at the rate of 30 miles per hour. Its 36-in. tire is revolving at the rate of how many radians per second?

70. A shaft rotates at the rate of 15,000 r.p.m. What is its angular velocity in radians per second?

71. Give the values of the circular functions of:

(*a*) 30°; (*b*) 60°; (*c*) 45°.

72. Give the algebraic signs of the functions of an angle in the first quadrant; in the second quadrant; in the third quadrant; in the fourth quadrant.

73. Give the functions of the following angles:

(*a*) 120°; (*b*) 135°; (*c*) 150°; (*d*) 210°; (*e*) 225°; (*f*) 240°; (*g*) 300°; (*h*) 315°; (*i*) 330°; (*j*) 0°; (*k*) 90°; (*l*) 180°; (*m*) 270°; (*n*) 360°.

74. Find the functions of α if:

(*a*) $\sin \alpha = \frac{3}{5}$ and $\cos \alpha$ be negative.
(*b*) $\sin \alpha = \frac{3}{5}$ and $\cos \alpha$ be positive.
(*c*) $\sin \alpha = \frac{3}{5}$ and $\tan \alpha$ be positive.
(*d*) $\sin \alpha = \frac{3}{5}$ and $\tan \alpha$ be negative.
(*e*) $\tan \alpha = 2$ and $\cos \alpha$ be negative.
(*f*) $\tan \alpha = -3$ and $\sin \alpha$ be positive.
(*g*) $\sec \alpha = 5$ and $\tan \alpha$ be negative.

75. Which of the circular functions are even functions? Which are odd functions?

76. Show that $\cos \alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$.

77. Draw the graph of $y = \sin x$.

78. Show that the curve for $y = \cos x$ may be obtained from the curve for $y = \sin x$ by translating it $\pi/2$ units to the left.

79. Show that $\sin^2 \alpha + \cos^2 \alpha = 1$.

80. Show that $\sec^2 \alpha = 1 + \tan^2 \alpha$.

81. Show that $\csc^2 \alpha = 1 + \cot^2 \alpha$.

82. Show that $\tan \alpha = \sin \alpha / \cos \alpha$.

83. Show that $\cot \alpha = \cos \alpha / \sin \alpha$.

84. Express $\sin x$ in terms of:

(a) $\cos x$; (b) $\tan x$; (c) $\cot x$; (d) $\sec x$; (e) $\csc x$.

85. Express $\cos x$ in terms of:

(a) $\sin x$; (b) $\tan x$; (c) $\cot x$; (d) $\sec x$; (e) $\csc x$.

86. Express $\tan x$ in terms of:

(a) $\sin x$; (b) $\cos x$; (c) $\cot x$; (d) $\sec x$; (e) $\csc x$.

87. The longer leg of a plot of land in the form of a 60° right triangle is 80 rods. Find the area of the plot in acres.

88. A plot of land in the form of a 60° right triangle contains 72 acres. Find the length in rods of each side of the triangle.

Hint: Let x represent the number of rods in the length of the shorter leg.

89. The shorter side of a rectangle is 100 ft. the diagonal is 200 ft. Find the length of the longer side.

90. Explain how points may be located in a plane by means of polar coordinates.

91. Define pole; polar axis; radius vector; vectorial angle.

92. Draw curves for:

(a) $\rho = 1$; (d) $\rho = 5$; (g) $\theta = \frac{\pi}{3}$;

(b) $\rho = 2$; (e) $\theta = 0$; (h) $\theta = \frac{\pi}{2}$;

(c) $\rho = 3$; (f) $\theta = \frac{\pi}{4}$; (i) $\theta = 2$.

Hint: θ is measured in radians.

93. What curve is represented by $\rho = a \cos \theta$? Prove.

94. What curve is represented by $\rho = b \sin \theta$? Prove.

95. Draw on a sheet of polar coordinate paper curves for the following:

(a) $\rho = 2 \cos \theta$. (c) $\rho = 2 \sin \theta$.

(b) $\rho = -2 \cos \theta$. (d) $\rho = -2 \sin \theta$.

96. Prove that $\rho = a \cos \theta + b \sin \theta$ is a circle.

97. Draw curves for the following:

(*a*) $\rho = 2\cos\theta + 3\sin\theta$. (*c*) $\rho = -2\cos\theta + \sin\theta$.
(*b*) $\rho = 3\cos\theta - 2\sin\theta$. (*d*) $\rho = -3\cos\theta - 3\sin\theta$.

98. Draw the circles $\rho = 1$ and $\rho = \cos\theta$ and from them plot the graph for $\rho = 1 + \cos\theta$.

99. Plot curves for the following equations:

(*a*) $\rho = 1 + \sin\theta$. (*c*) $\rho = 2 + \cos\theta$.
(*b*) $\rho = 1 - \sin\theta$. (*d*) $\rho = 1 - 2\cos\theta$.

100. From a sheet of polar coordinate paper (form *M*3) find values for the following:

(*a*) sin 30°. (*b*) cos 30°. (*c*) sin 45°. (*d*) cos 45°. (*e*) tan 45°. (*f*) cos 10°. (*g*) sin 116°. (*h*) cos 216°. (*i*) sin 127°. (*j*) tan 37°. (*k*) sin 227°. (*l*) cos 316°.

101. Show that when the curve for $\rho = f(\theta)$ is rotated about the pole through an angle α, its equation becomes $\rho = f(\theta - \alpha)$.

102. State nine "Theorems on Loci."

103. Find the polar equation of a straight line.

104. The center of the circle $\rho = 10\sin(\theta - \alpha)$ lies on the line $x - y = 3$. Find α.

105. The center of the circle $\rho = 10\cos(\theta - \alpha)$ lies on the line $3x - 2y = 1$. Find α.

106. The center of the circle $\rho = 5\sin(\theta + \alpha)$ lies on the line $x - 2y = 6$. Find α.

107. Write the Cartesian equations for:

(*a*) $\rho = 2\cos\theta + 3\sin\theta$.
(*b*) $\rho = 3\cos\theta - 2\sin\theta$.
(*c*) $\rho = 2\sin\theta - 3\cos\theta$.

108. Solve analytically $2 = 2\cos\theta - 3\sin\theta$ for all values of θ between 0° and 360°.

109. Solve graphically the equation given in Exercise 108.

110. Sketch a curve for $y = 1/x - 2x$.

111. Sketch a curve for $y = x/2 + \sin x$.

112. A circle is inscribed in a 30°, 60° right triangle. Find the diameter of the circle if the shorter leg of the triangle be 4 in.; if the longer leg of the triangle be 6 in.; if the hypotenuse of the triangle be 10 in. Find the lengths of the three sides of the triangle if the radius of the inscribed circle be 6 in.

113. A circle is inscribed in a 45° right triangle. Find the diameter of the circle if the legs of the triangle be each 4 in. in length.

114. A circle is circumscribed about a 30°, 60° right triangle. Find the radius of the circle if the hypotenuse of the triangle be 10 in.

115. Write the polar equation for

$$x^2 - y^2 = a^2(x^2 + y^2)^2.$$

116. Define an ellipse; major axis; minor axis.

117. Give parametric equations for an ellipse.

118. Find the coordinates of the center and the lengths of the semi-axes of the ellipse

$$x = 3 + 2\cos\phi,$$
$$y = 2 - \sin\phi.$$

119. Show that every section of a right circular cylinder by a plane is an ellipse.

120. Show that the projection of a circle upon a plane is an ellipse.

121. Define an equilateral, or rectangular, hyperbola. Define an hyperbola.

122. Give parametric equations for an hyperbola.

123. Define the axes, the center, and the asymptotes of an hyperbola.

124. Find the coordinates of the center, the lengths of the semi-axes, and the equations of the asymptotes for

$$x^2 - y^2 + 2x - 4y = 11.$$

125. Find the equation for the curve of $xy = 4$ when rotated about the origin through an angle of $-45°$.

126. Define conjugate hyperbolas.

127. Write the equation of the hyperbola conjugate to

$$x^2 - y^2 - x + 4y = 11.$$

128. Sketch the curve with asymptotes for

$$x = 3 + 2\sec\phi$$
$$y = 1 - 3\tan\phi.$$

129. Write the equation of the curve formed when the circle $x^2 + y^2 = a^2$ is sheared in the line $y = x$. Sketch the curve.

130. Write the equation of the curve formed when the hyperbola $x^2/4 - y^2/9 = 1$ is sheared in the line $y = x$. Sketch the curve.

131. State and prove the remainder theorem.

132. State and prove the factor theorem.

133. Without performing the division, find the remainder of $(x^3 - 2x^2 + 3x - 1) \div (x + 2)$.

134. Explain what is meant by questionable and legitimate transformations.

135. Explain a method of finding approximately the roots of a cubic equation.

136. Find the equation of the straight line passing through the points of intersection of $x^2 + y^2 + 2x + 4y - 11 = 0$ and $x^2 + y^2 - 2x - 2y = 0$.

137. What are the equations of the coordinate axes?

138. What is the locus of $x^2 = 4$? Of $y^2 = 4$? Of $x^2 = y^2$? Of $a^2x^2 = b^2y^2$?

139. Solve $\left\{\begin{matrix} 7x^2 - 6xy = 8 \\ 2x - 3y = 5 \end{matrix}\right\}$.

140. Solve $\left\{\begin{matrix} 4x^2 - 25y^2 = 0 \\ 2x^2 - 10y^2 - 3y = 4 \end{matrix}\right\}$.

141. Define series.

142. Define arithmetical progression; geometrical progression.

143. Define arithmetical mean; geometrical mean.

144. Derive formulas for l and s of an arithmetical progression.

145. Derive formulas for l and s of a geometrical progression.

146. Define an infinite geometrical progression.

147. Derive the formula for the sum of an infinite geometrical progression.

148. Find the value of 0.273273273 . . .

149. A debt of \$10,000 is to be paid in 10 years. An equal amount is paid at the end of each year. Find this amount if the indebtedness draws interest at 5 per cent.

150. An equal amount of money is deposited at the end of each year for 20 years as a sinking fund to replace a piece of machinery valued at \$10,000. How much must be deposited at the end of each year, if the deposits draw 4 per cent compound interest.

151. What curves are represented by the equations $\rho = 2\cos(\theta - \pi/6)$ and $2 = \rho\cos(\theta - \pi/6)$? How are these curves related to each other? Sketch the curves.

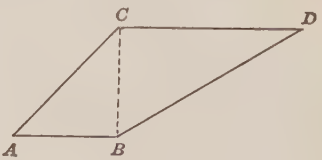


FIG. 90.

152. Find the perimeter and area of the figure $ABCD$ if $AB = 100$ ft., $\angle BAC = \angle BCA = 45°$, $\angle CDB = \frac{1}{2}\angle CBD = 30°$. (See Fig. 90.)

153. Each of the sides of a rhombus is 120 ft. What is the length in feet of the short diagonal if the acute angle of the rhombus be 60°? What is the length of the short diagonal if the acute angle be 40°? Check.

154. Prove that if a circular cylinder be cut by a plane at an angle of 45° to the axis of the cylinder, and if then the surface of the cylinder be unrolled into a flat surface, the curved boundary of the surface is a sinusoid. Thus, if a stove pipe be cut at an angle of 45° to its axis, and if then the sheet metal be unrolled into a flat sheet, the bounding curve is a sinusoid.

In Fig. 91 only one quarter of the cylinder is shown. If P be any point on the section of the cylinder made by the cutting plane, and if

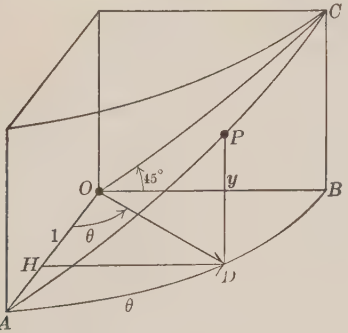


FIG. 91.—Diagram for Exercise 154.

the length of the arc AD be called θ and the distance DP be called y, the problem is to show that $y = \sin\theta$, provided the radius of the cylinder be called 1. If the angle of cutting be different from 45°, the equation of the curve is of the form $y = b \sin\theta$, where $b = \tan BOC$.

155. A traveler's rule for determining the distance one can see from a given height above a level surface of the earth (such as a plain or the sea) is as follows: "To the height in feet add half the height and take the square root. The result is the distance you can see in miles." Show that this result is approximately correct, assuming the earth a sphere of radius 3960 miles. Show that the drop in 1 mile is 8 in., and that the water in the middle of a lake 8 miles in width stands $10\frac{2}{3}$ ft. higher than the water at the shores.

CHAPTER VIII

THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS

114. Historical Development.—The almost miraculous power of modern calculation is due, in large part, to the invention of logarithms in the first quarter of the seventeenth century by a Scotchman, John Napier, Baron of Merchiston. This invention was founded on a very simple and obvious principle, which had been quite overlooked by mathematicians for many generations. Napier's invention may be explained as follows.[1] Let there be an arithmetical and a geometrical progression which are to be associated together, as, for example, the following:

$$\begin{array}{cccccccccccc} 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10 \\ 1, & 2, & 4, & 8, & 16, & 32, & 64, & 128, & 256, & 512, & 1024 \end{array} \quad \text{(I)}$$

Now the product of any two numbers of the second line may be found by adding the two numbers of the progression above them, finding this sum in the first line, and, finally, taking the number lying under it; this latter number is the product sought. Thus suppose the product of 8 by 32 be desired. Over these numbers of the second line stand the numbers 3 and 5, whose sum is 8. Under 8 is found 256, the product desired. Now since only a limited variety of numbers is offered in this table, it would be useless in the actual practice of multiplication, for the reason that the particular numbers whose product is desired would probably not be found in the second line. The overcoming of this obvious obstacle constitutes the novelty of Napier's invention. Napier proposed to insert any number of intermediate terms in each progression. Thus, instead of the portion

$$\begin{array}{ccccc} 0, & 1, & 2, & 3, & 4 \\ 1, & 2, & 4, & 8, & 16 \end{array}$$

[1] Merely the fundamental principles of the invention, not historical details, are given in what follows.

of the two series, we may write

$$\begin{array}{cccccccccc} 0, & \frac{1}{2}, & 1, & 1\frac{1}{2}, & 2, & 2\frac{1}{2}, & 3, & 3\frac{1}{2}, & 4 \\ 1, & \sqrt{2}, & 2, & \sqrt{8}, & 4, & \sqrt{32}, & 8, & \sqrt{128}, & 16 \end{array}$$

by inserting arithmetical means between the consecutive terms of the arithmetical series and by inserting geometrical means between the terms of the geometrical series. Let these be computed to any desired degree of accuracy, say to two decimal places. Then we have the series

A. P.	G. P.	
0.0	1.00	
0.5	1.41	
1.0	2.00	
1.5	2.83	(II)
2.0	4.00	
2.5	5.66	
3.0	8.00	

Again, inserting arithmetical and geometrical means between the terms of the respective series we have:

A. P.	G. P.	
0.00	1.00	
0.25	1.19	
0.50	1.41	
0.75	1.69	
1.00	2.00	(III)
1.25	2.38	
1.50	2.83	
1.75	3.36	
2.00	4.00	
2.25	4.76	

By continuing this process each consecutive three-figure number may finally be made to appear in the second column, so that, to this degree of accuracy, the product of any two such numbers may be found by the process previously explained. The decimal points of the factors may be ignored in this work, as, for example, the product of 2.38×14.1 is the same as that of 238×14.1, except in the position of the decimal point. The correct position of the decimal point can be determined by inspection after the

significant figures of the product have been obtained. Using the above table we find $2.38 \times 14.1 = 33.6$.

The above table, when properly extended, is a **table of logarithms.** The first column of figures contains the **logarithms** of the numbers that stand opposite them in the second column. Napier, by this process, said he divided the ratio of 1.00 to 2.00 into "100 equal ratios," by which he referred to the insertion of 100 geometrical means between 1.00 and 2.00. The "*number of the ratio*" he called the **logarithm** of the number—for example, 0.75 opposite 1.69, is the logarithm of 1.69. The word "logarithm" is from two Greek words meaning "*the number of the ratios.*" In order to produce a table of logarithms, it was merely necessary to compute numerous geometrical means; that is, no operations except multiplication and the extraction of square roots were required. But the numerical work was carried out by Napier to so many decimal places that the computation was exceedingly difficult.

In (I) the geometrical progression has a ratio of 2, and is so placed that its term 1 corresponds to the term 0 in the arithmetical progression. The student should satisfy himself, by constructing a similar table in which the geometrical progression has a ratio of, say, $\frac{1}{2}$ that as long as the term 1 of the geometrical progression corresponds with the term 0 of the arithmetical progression (see Exercise 1 below) the multiplication scheme works as explained above, and is not dependent upon the value of the ratio. Thus the number of possible systems of logarithms is indefinitely great.

Reread Sec. **3** which contains the definition of a function. The progressions printed in tabular form above *define a function*, each number in one progression being a function of the number in the associated progression. The function thus defined will be given a name in the next section.

The news of the remarkable invention of logarithms induced Henry Briggs, professor at Gresham College, London, to visit Napier in 1615. It was on this visit that Briggs suggested the advantages of a system of logarithms in which the logarithm of 10 should be 1, for then it would only be necessary to insert a sufficient number of geometrical means between 1 and 10 to obtain the logarithm of any desired number. With the encourage-

ment of Napier, Briggs undertook the computation, and in 1617 published the logarithms of numbers from 1 to 1000, and in 1624 the logarithms of numbers from 1 to 20,000, and from 90,000 to 100,000 to *fourteen decimal places.* The gap between 20,000 and 90,000 was filled by a Hollander, Adrian Vlacq, whose table, published in 1628, is the source from which nearly all the tables since published have been derived.

Exercises

1. By means of the progressions (I), explain how to multiply 16 by 1. Why in a pair of progressions to be used for multiplication purposes must the term 1 of the geometrical progression correspond to the term 0 of the arithmetical progression?

2. Construct a table similar to (I), but use 3 for the ratio of the geometrical progression. Let the arithmetical progression run from -12 to $+12$. Use these progressions to compute the product of $\frac{1}{9}$ by 243, and of 3 by 27.

3. Consider a geometrical progression having the ratio 2. If means are inserted between every two terms, what is the ratio of the new geometrical progression? If this process is repeated again, what is the ratio? What is the ratio after the process has been repeated k times? This problem should make it clear that, regardless of how many means are inserted, a system of logarithms based on a geometrical progression whose ratio is 2 is essentially different from a system based on a geometrical progression whose ratio is, say, 3.

115. The Exponential Function.—In the original pair of progressions (I) of the preceding section, let any number in the arithmetical progression be denoted by x, and any number of the geometrical progression by y. Then, according to definition,

$$x = \log y.$$

To indicate that the system of logarithms in question is based upon a geometrical progression whose ratio is 2, this equation may be written

$$x = \log_2 y,$$

which is read "x is the logarithm of y to the base 2," the ratio of the geometrical progression being called the **base** of the system of logarithms. Similarly, the equation

$$x = \log_a y \tag{1}$$

indicates that x is the logarithm of y to the base a.

The relationship between the numbers x of the arithmetical progression and the numbers y of the geometrical progression can, however, be expressed in another form. By examining the progression (I) it is seen that any number y in the geometrical progression is the xth power of 2 where x is the corresponding number in the arithmetical progression. For example, 128 is the seventh power of 2, 1024 the tenth power, etc. Thus,

$$y = 2^x.$$

This equation holds not only for the progression (I) but also for the progressions formed from (I) by the repeated insertion of arithmetical and geometrical means. This statement can be verified by inspection of the progressions (II) and (III) above, and proved in general by the result of Exercise 3 of the preceding section. If the ratio of the geometrical progression had been a rather than 2, this equation would read

$$y = a^x. \tag{2}$$

Thus, *when the values of a variable x run over an arithmetical progression (of first term 0) while the corresponding values of a variable y run over a geometrical progression (of first term 1), the relation between the variables may be written in either of the forms* (1) *or* (2).

Equation (2) is called an **exponential equation,** and y is said to be an **exponential function** of x, while in Eq. (1) x is said to be a **logarithmic function** of y. To emphasize the fact that the base is a, the number y is sometimes called the exponential function of x to the base a, and is sometimes written $y = \exp_a x$. The student has frequently been called upon in mathematics to express relations between variables in two different or inverse forms, analogous to the two forms $y = a^x$ and $x = \log_a y$. For example he has written either

$$y = x^2, \text{ or } x = \pm\sqrt{y};$$

either

$$y = x^{n/2}, \text{ or } x = y^{2/n};$$

and either

$$y = \cos x, \text{ or } x = \text{arc cos } y.$$

The graph of a functional relationship is, of course, the same whether the equation expressing this relationship be solved for x or solved for y.

Exercises

1. Write the following equations in logarithmic form

(*a*) $y = 10^x$. (*d*) $u = 5^t$.
(*b*) $y = 3^x$. (*e*) $z = a^u$.
(*c*) $y = a^x$. (*f*) $u = 1.1^s$.

2. Write the following equations in exponential form:

(*a*) $x = \log_{10} y$. (*d*) $u = \log_a y$.
(*b*) $x = \log_3 y$. (*e*) $t = \log_5 z$.
(*c*) $x = \log_2 y$. (*f*) $u = \log_6 x$.

3. Find the values of the following:

(*a*) $\log_{10} 100$. (*d*) $\log_2 64$.
(*b*) $\log_5 25$. (*e*) $\log_3 81$.
(*c*) $\log_3 27$. (*f*) $\log_{\sqrt{2}} 16$.

4. In the last chapter the general term of any geometrical progression was written ar^{n-1}. Show that this reduces to Eq. (2) above for progressions like (I), Sec. **114.**

116. Graphic Computation of the Terms of a Geometrical Progression.—Draw the line $y = x$ and $y = ax$ (Fig. 92). From

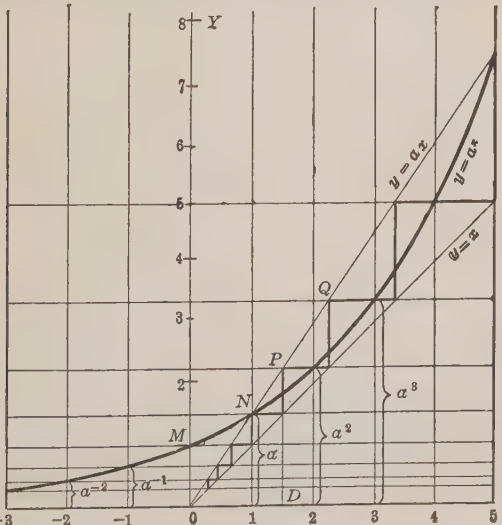


FIG. 92.—Graphical construction of the successive terms of a *G. P.* In the diagram $a = \frac{3}{2}$, and the curve is $y = (\frac{3}{2})^x$.

the point $(1, a)$ on $y = ax$ draw a horizontal line to $y = x$, thence a vertical line to $y = ax$, etc., thereby forming the "stairway" of line segments between $y = x$ and $y = ax$ as shown in Fig. 92. Then the points N, P, Q, etc. have the ordinates a, a^2, a^3, etc., as required; for, to obtain the ordinate PD of P, the value of x used was $OD = a$; hence P is the point on $y = ax$ for $x = a$, or $y = PD = a^2$. Likewise, Q is, by construction, the point on $y = ax$ for $x = a^2$; hence the y of the point $Q = a \cdot a^2 = a^3$, etc. The ordinates of the "treads" of the "stairway" thus formed are terms of the geometrical progression $1, a, a^2, a^3, \ldots$ and their intersections with the vertical lines $x = 0$, $x = 1$, $x = 2$, $x = 3$, . . . are points on the curve $y = a^x$ or $x = \log_a y$.

The method just described locates the points determined by pairs of corresponding values in the progression

$$\begin{array}{cccccc} 0, & 1, & 2, & 3, & 4, & \ldots \\ 1, & a, & a^2, & a^3, & a^4, & \ldots \end{array}$$

Suppose it be desired to insert arithmetical and geometrical means, as follows:

$$\begin{array}{ccccccccc} 0, & \frac{1}{2}, & 1, & \frac{3}{2}, & 2, & \frac{5}{2}, & 3, & \frac{7}{2}, & 4 \ldots \\ 1, & a^{\frac{1}{2}}, & a, & a^{\frac{3}{2}}, & a^2, & a^{\frac{5}{2}}, & a^3, & a^{\frac{7}{2}}, & a^4 \ldots \end{array}$$

and to locate, on the graph, the points which arise from these pairs of corresponding numbers. The procedure is clearly the same as before, except that, since the ratio of the geometrical progressions is now $a^{\frac{1}{2}}$, we use the line $y = x$ and $y = \sqrt{ax}$, the points being located by the intersections of the horizontal treads and the vertical lines $x = 0$, $x = \frac{1}{2}$, $x = 1$, $x = \frac{3}{2}$, $x = 2$, etc. It is clear that in this manner we obtain all the points obtained before, and also new intermediate ones. By using the lines $y = x$ and $y = \sqrt[4]{a\,x}$, the number of points may again be doubled, and so on, so that the curve may be drawn as accurately as desired. However, in constructing a stairway an error in the position of any tread or riser affects the position of every subsequent tread and riser. It is advisable, therefore, to locate a set of points whose abscissas are $x = 0, 1, 2, 3, \ldots$ by use of the lines $y = x$ and $y = ax$, and then use the line $y = \sqrt{a\,x}$ to obtain only the intermediate points $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ thus avoiding an accumulation of errors.

Figure 93 shows the result of a graphic construction of the curve $y = 2^x$. The first stairway, namely, that between the lines $y = x$ and $y = 2x$, is drawn in heavy lines in the figure. It is unnecessary, in determining the intermediate points, actually to construct a second stairway between the lines $y = x$ and $y = \sqrt{2}\,x$, since the treads of this second stairway which locate the intermediate points

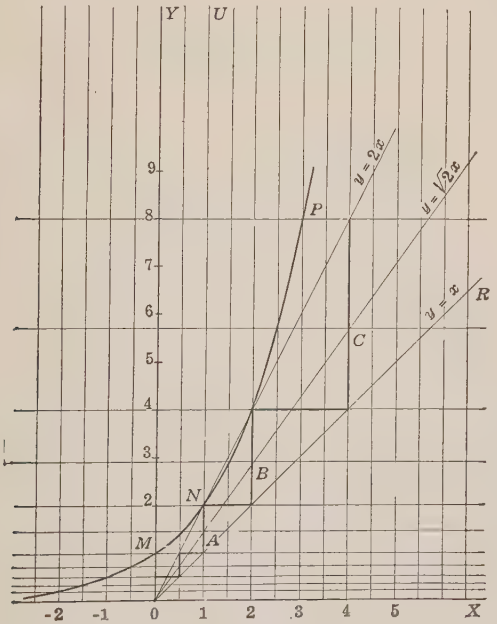


FIG. 93.—Graphical construction of the curve $y = 2^x$.

pass through the intersections of the line $y = \sqrt{2}\,x$ and the riser of the first stairway. The student should draw for himself a sketch on which both stairways appear, and convince himself of the truth of this statement. The actual procedure in Fig 93 is, therefore, to draw vertical lines through $x = 0$, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, etc., and horizontal lines through the treads of the first stairway and through the points in which the line $y = \sqrt{2}\,x$ cuts the risers of this stairway. This process divides the plane into a large number of rectangles.

Starting at the point M of coordinates (0, 1) and sketching the diagonals (curved to fit the alignment of the points) of successive "cornering" rectangles, the desired graph is obtained: If it be desired to construct this graph still more accurately, the number of located points may again be doubled by making use of the line $y = \sqrt[4]{2}\,x$, etc.

Exercise

If the line $y = \sqrt[4]{2}\,x$ is to be used, show that the stairway between $y = x$ and $y = 2x$ and the stairway between $y = x$ and $y = \sqrt{2}x$ must be constructed, but that the stairway between $y = x$ and $y = \sqrt[4]{2}\,x$ need not be.

117. Common Logarithms.—In the equation $10^x = y$, x is called the **common logarithm of y.** It is also called the **Brigg's logarithm of y.** Thus, the common logarithm of any number is the exponent of the power to which 10 must be raised to produce the given number. Thus 2 is the common logarithm of 100, since $10^2 = 100$; likewise, 1.3010 will be found to be the common logarithm of 20 correct to four decimal places, since $10^{1.3010} = 20.00$.

118. Systems of Logarithms.—If in the exponential equation $y = a^x$, where a is any positive number except 1, different values be assigned to y and the corresponding values of x be computed and tabulated, the results constitute a **system of logarithms.** The number of different possible systems is unlimited, as already noted in Sec. **114.** As a matter of fact, however, only two systems have been computed and tabulated; the **natural,** or **Naperian,** or **hyperbolic, system,** whose base is the incommensurable number e, approximately 2.7182818, and the **common,** or **Brigg's, system,** whose base is 10. The letter e is set aside in mathematics to stand for the base of the natural system.

Natural logarithms of all numbers from 1 to 20,000 have been computed to 17 decimal places. The common logarithms are usually printed in tables of four, five, six, seven or eight decimal places.

It will be found later that the graphs of all logarithmic functions of the form $x = \log_a y$ can be made by stretching or by contracting in the same fixed ratio the ordinates of any one of the logarithmic curves. That is, the logarithms of one system can be

obtained from those of another system by multiplying by a constant factor. For this reason numerical tables in more than one system of logarithms are unnecessary.

In the following pages the common logarithm of any number n will be written $\log n$, and not $\log_{10} n$; that is, the base is supposed to be 10 unless otherwise designated; $\ln x$ for $\log_e x$ and $\lg x$ for $\log_{10} x$ are also used.

Exercises

Write the following in logarithmic notation:

1. $10^3 = 1000.$
2. $10^{-3} = 0.001.$
3. $10^0 = 1.$
4. $11^2 = 121.$
5. $16^{0.25} = 2.$
6. $e^x = y.$
7. $10^{0.25} = 1.7783.$
8. $10^{0.3010} = 2.$
9. $a^1 = a.$
10. $10^{\log_{10} y} = y.$

Express the following in exponential notation:

11. $\log_{10} 4 = 0.6021.$
12. $\log 10000 = 4.$
13. $\log 0.0001 = -4.$
14. $\log_2 1024 = 10.$
15. $\log_a a = 1.$
16. $\log \sqrt[3]{100} = \frac{2}{3}.$
17. $\log_{27} (\frac{1}{3}) = -\frac{1}{3}.$
18. $\log_{100} 10 = \frac{1}{2}.$
19. $\log 1 = 0.$
20. $\log_a 1 = 0.$

119. Graphic Table; Characteristic and Mantissa.—In Fig. 94 is shown the graph of the function defined by the two progressions whose use was suggested by Briggs to Napier, and which are referred to in the last paragraph of Sec. **114.** By inserting means three times between 0 and 1 in the arithmetical progression and between 1 and 10 in the geometrical progression, we get

A. P. or logarithms	G. P. or numbers	Exponential form of G. P.
0.000	1.000	$10^{0.000}$
0.125	1.334	$10^{0.125}$
0.250	1.778	$10^{0.250}$
0.375	2.371	$10^{0.375}$
0.500	3.162	$10^{0.500}$
0.625	4.217	$10^{0.625}$
0.750	5.623	$10^{0.750}$
0.875	7.499	$10^{0.875}$
1.000	10.000	$10^{1.000}$

If we let L stand for the logarithm of the number N, the functional relation is, obviously, $L = \log_{10} N$, or $N = 10^L$. The curve (Fig. 94) may now be used as a graphic table of logarithms from which the results can be read to about two decimal places. The logarithms of numbers between 1 and 10 may be read directly from the graph. Thus, $\log_{10} 7.24 = 0.860$. If the logarithm be between 0 and 1, the number is read directly from the graph. Thus if the logarithm be 0.273, the number is 1.87.

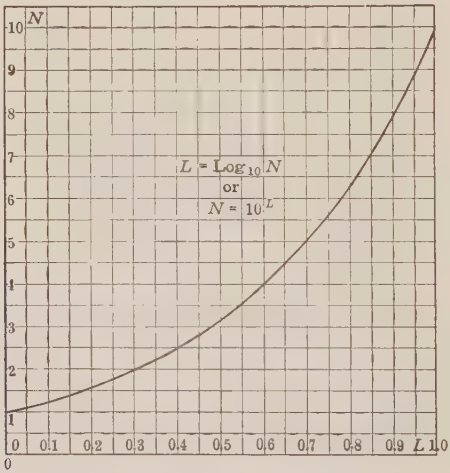


FIG. 94.—The curve $L = \log_{10} N$.

If we multiply the readings of the N-scale by 10^n, we must add n to the readings on the L-scale, for $10^n N = 10^{L+n}$.

If we divide the readings on the N-scale by 10^n, we must subtract n from the readings on the L-scale, for $N/10^n = 10^{L-n}$.

Thus we have:

$$\begin{aligned} \log 7.24 &= 0.860 \\ \log 72.4 &= 1.860 \\ \log 724 &= 2.860 \\ \log 7240 &= 3.860 \end{aligned}$$

and also

$$\log 0.724 = 0.860 - 1$$
$$\log 0.0724 = 0.860 - 2$$
$$\log 0.00724 = 0.860 - 3.$$

Now the common logarithm of a number is always written so that it consists of a positive decimal part and an integral part which may be either positive or negative; that is to say, the logarithm of, for example, 0.724 is not written -0.140 (although it is equal to this negative number), but is written $0.860 - 1$.

When a logarithm of a number is thus arranged, special names are given to each part. The positive or negative integral part is called the **characteristic** of the logarithm. The *positive* decimal part is called the **mantissa.** Thus, in $\log 200 = 2.3010$, the characteristic is 2, and 3010 is the mantissa. In $\log 0.02 = 0.3010 - 2$, the characteristic is -2, and 3010 is the mantissa.

An examination of the logarithms written above shows that: (1) the value of the characteristic is dependent merely upon the position of the decimal point in the number; (2) the value of the mantissa is the same for the logarithms of all numbers that differ only in the position of the decimal point. In particular, we derive therefrom the following rule for finding the characteristic of a common logarithm of any number:

The characteristic of the common logarithm of a number equals the number of places the first significant figure of the number is removed from units' place, and is positive if the first significant figure stand to the left of units' place and is negative if it stand to the right of units' place.

Thus, in $\log 1910 = 3.2810$, the first figure 1 is three places from units' place and the characteristic is 3. In $\log 0.0191 = 0.2810 - 2$, the first significant figure 1 is two places to the right of units' place and the characteristic is -2. A computer in determining the characteristic of the logarithm of a number first points to units' place and counts zero, then passes to the next place and counts one, and so on until the first significant figure is reached.

In computing with logarithms, it is almost universal practice to write logarithms with negative characteristics, such as $0.3010 - 1$, $0.3010 - 2$, etc., in the equivalent form $9.3010 - 10$, $8.3010 - 10$,

The characteristics $9 - 10$, $8 - 10$, etc. can be obtained from the above rule, or from any equivalent form which the student may care to devise.

120. Properties of Logarithms.—The following properties of logarithms follow at once from the general properties or laws of exponents.

1. *The logarithm of* 1 *is* 0 *in all systems,* for $a^0 = 1$, that is, $\log_a 1 = 0$. In Fig. 93, note that the curve passes through (0, 1).

2. *The logarithm of the base itself in any system is* 1, for $a^1 = a$, that is, $\log_a a = 1$. In Fig. 93, by construction N is always the point $(1, a)$, where a is the ratio of the first or fundamental progression in which means are inserted.

3. *Negative numbers have no logarithms.* This follows at once from Fig. 93. In Figs. 90, 93 and 94 note that the curves do not extend below the X-axis.

NOTE: While negative numbers have no logarithms, this does not prevent the computation of expressions containing negative factors and divisors. Thus, to compute $(287) \times (-374)$, find $(287) \times (374)$ by logarithms and give proper sign to the result.

121. Logarithm of a Product.—Let n and r be any two positive numbers, and let

$$\log_a n = x \text{ and } \log_a r = y. \tag{1}$$

Then, by definition of a logarithm (Sec. **115**),

$$n = a^x \text{ and } r = a^y. \tag{2}$$

Hence,

$$nr = a^x a^y = a^{x+y}.$$

Therefore, by definition of a logarithm (Sec. **115**),

$$\log_a nr = x + y,$$

or, by Eq. (1),

$$\log_a \mathbf{nr} = \log_a \mathbf{n} + \log_a \mathbf{r}. \tag{3}$$

Hence, *the logarithm of the product of two numbers is equal to the sum of the logarithms of those numbers.*

In the same way, if $\log_a s = z$, then

$$nrs = a^{x+y+z},$$

that is,

$$\log_a \mathbf{nrs} = \log_a \mathbf{n} + \log_a \mathbf{r} + \log_a \mathbf{s}.$$

Exercises

Find the results by the formulas and check by the curve of Fig. 94.

1. Given $\log 2 = 0.3010$, and $\log 3 = 0.4771$; find $\log 6$; find $\log 18$.

2. Given $\log 5 = 0.6990$ and $\log 7 = 0.8451$; find $\log 35$.

3. Given $\log 9 = 0.9542$, find $\log 81$.

4. Given $\log 386 = 2.5866$ and $\log 857 = 2.9330$; find the logarithm of their product.

5. Given $\log 11x = 1.888$ and $\log 11 = 1.0414$; find $\log x$.

122. Logarithm of a Quotient.—Let n and r be any two positive numbers, and let

$$\log_a n = x \text{ and } \log_a r = y. \tag{1}$$

From Eq. (1), by the definition of a logarithm,

$$n = a^x \text{ and } r = a^y.$$

Hence,

$$\frac{n}{r} = a^x \div a^y = a^{x-y}.$$

Therefore, by definition of a logarithm,

$$\log_a \left(\frac{n}{r}\right) = x - y,$$

or by Eq. (1)

$$\log_a \left(\frac{\mathbf{n}}{\mathbf{r}}\right) = \log_a \mathbf{n} - \log_a \mathbf{r}. \tag{2}$$

Hence, *the logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

Exercises

Find the results by the formulas and check by the curve of Fig. 94.

1. Given $\log 5 = 0.6990$ and $\log 2 = 0.3010$; find $\log (5/2)$; find $\log 0.4$.

2. Given $\log 63 = 1.7993$, and $\log 9 = 0.9542$; find $\log 7$.

3. Given $\log 84 = 1.9243$ and $\log 12 = 1.0792$; find $\log 7$.

4. Given $\log 1776 = 3.2494$ and $\log 1912 = 3.2815$; find

$\log 1776/1912$; find $\log 1912/1776$.

123. Logarithm of Any Power.—Let n be any positive number and let

$$\log_a n = x. \tag{1}$$

From Eq. (1), by the definition of a logarithm,

$$n = a^x.$$

Raising both sides to the pth power, *where p is any number whatsoever*,

$$n^p = a^{px}.$$

Therefore, by definition of a logarithm,

$$\log_a (n^p) = px,$$

or, by Eq. (1),

$$\log_a (n^p) = p \log_a n. \tag{2}$$

Hence, *the logarithm of any power of a number equals the logarithm of the number multiplied by the index of the power.*

The above includes the two cases: (1) the finding of the logarithm of any integral power of a number; since, in this case p is a positive integer; and (2) the finding of the logarithm of any root of a number, since in this case p is the reciprocal of the index of the root.

Exercises

1. Given $\log 2 = 0.3010$; find $\log 1024$; find $\log \sqrt{2}$; find $\log \sqrt[5]{2}$.

2. Given $\log 1234 = 3.0913$; find $\log \sqrt{1234}$; find $\log \sqrt[5]{1234}$.

3. Given $\log 5 = 0.6990$; find $\log 5^{\frac{2}{3}}$; find $\log 5^{\frac{3}{2}}$.

4. Show that $\log (\frac{11}{15}) + \log (\frac{490}{297}) - 2 \log (\frac{7}{9}) = \log 2$.

5. Find an expression for the value of x from the equation $3^x = 567$.

SOLUTION: Take the logarithm of each side;

$$x \log 3 = \log 567.$$

But

$$\log 567 = \log (3^4 \times 7) = 4 \log 3 + \log 7.$$

Therefore

$$x \log 3 = 4 \log 3 + \log 7,$$

or

$$x = 4 + \left(\frac{\log 7}{\log 3}\right).$$

6. Find an expression for x in the equation $5^x = 375$.

7. Given $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find how many digits in 6^{10}.

8. Find an expression for x from the equation

$$3^x \times 2^{x+1} = \sqrt{512}.$$

△ **9.** Prove that $\log (7\frac{5}{16}) - 2 \log (5\frac{5}{9}) + \log (3\frac{2}{243}) = \log 2$.

124. Logarithmic Tables.—A table of common logarithms contains only the *mantissas* of the logarithms of a certain convenient sequence of numbers. For example, a four-place table will contain the mantissas of the logarithms of numbers from 100 to 1000; a five-place table will usually contain the mantissas of the logarithms of numbers from 1000 to 10,000, and so on. Of course, it is unnecessary to print decimal points or characteristics.

A table of logarithms should contain means for readily obtaining the logarithms of numbers intermediate to those tabulated, by means of **tabular differences** and **proportional parts.**[1]

The tabular differences are the differences between successive mantissas. If any tabular difference be multiplied successively by the numbers 0.1, 0.2, 0.3, . . . 0.8, 0.9, the results are called the **proportional parts.** Thus, from a four-place table we find $\log 263 = 2.4200$. The tabular difference is given in the table as 16. If we wish the logarithm of 263.7, the proportional part 0.7×16, or 11.2, is added to the mantissa, giving, to four places, $\log 263.7 = 2.4211$. This process is known as **interpolation.** Corrections of this kind are made with great rapidity after a little practice. It is obvious that the principle used in the correction is the equivalent of a geometrical assumption that the graph of the function is *nearly straight* between the successive values of the argument given in the table. *The corrections should* **invariably be added mentally** *and all the work of interpolation should be done mentally if the finding of the proportional parts by mental work does not require multiplication beyond the range of* 12×12. *To make interpolations mentally* **is an essential practice,** if the student is to learn to compute by logarithms with skill.

A good method is the following: Suppose $\log 13.78$ is required. First write down the characteristic 1; then, with the table at your left, find 137 in the number column and mark the corresponding mantissa by placing your thumb above it or your first finger

[1] The student is supposed to have SLICHTER'S "Four Place Tables," The Macmillan Co., New York. The edition printed on three sheets of heavy manilla paper perforated to fit in notebook is preferred. See also tables at end of this book.

below it. *Do not read this mantissa,* but read the tabular difference, 32. From the proportional parts table find the correction, 26, for 8. Now return to the mantissa marked by your finger, and read it increased by 26, *i.e.*, 1393; then place 1393 after the characteristic 1 previously written down.

The accuracy required for nearly all engineering computations does not exceed three or four significant figures. Four-figure accuracy means that the errors permitted do not exceed 1 *per cent of* 1 *per cent.* Only a small portion of the fundamental data of science is reliable to this degree of accuracy.[1] The usual measurements of the testing laboratory fall far short of it. Only in certain work in geodesy, and in a few other special fields of engineering, need more than four-place logarithms be used.

Exercises

Find the logarithms of the following:

1. 136.	**4.** 375.5.	**7.** 2.758.
2. 752.	**5.** 217.6.	**8.** 762,700.
3. 976.	**6.** 17.62.	**9.** 0.1278.

125. Anti-logarithms.—If we wish to find the number which has a given logarithm, it is convenient to have a table in which the logarithm is printed *before* the number. Such a table is known as a table of **anti-logarithms.** It is usually not best to print tables of anti-logarithms to more than four places; to find a number when a five-place logarithm is given, it is preferable to use the table of logarithms inversely, as the large number of pages required for a table of anti-logarithms is a disadvantage which is not compensated for by the additional convenience of such a table.

Exercises

From a four-place table of anti-logarithms, find the numbers corresponding to the following logarithms:

1. 2.7864.	**3.** 1.8152.	**5.** 8.1278 − 10.
2. 3.1286.	**4.** 9.6278 − 10.	**6.** 6.1785 − 10.

[1] Fundamental constants upon which much of the calculation in applied science must be based are not often known to four figures. The mechanical equivalent of heat is hardly known to 1 per cent. The specific heat of superheated steam is even less accurately known. The tensile, tortional, and compressive strength of no structural material would be assumed to be known to a greater accuracy than the above-named constants. Of course, no calculated result can be more accurate than the least accurate of the measurements upon which it depends.

126. Cologarithms.—Any computation involving multiplication, division, evolution, and involution may be performed by the addition of a single column of logarithms. This possibility is secured by noting the fact that subtracting $\log n$ (that is, dividing by n) is equivalent to adding $-\log n$. When the negative of the logarithm of a number is written in the form $(10 - \log n) - 10$, it is called the **cologarithm** of the number.

The convenience arising from this use may be illustrated as follows: Suppose it be required to find x from the proportion

$$37.4^2 : x :: 647 : \sqrt{0.582}.$$

We then have

$$\begin{aligned} 2 \log 37.4 &= 3.1458 \\ \tfrac{1}{2} \log 0.582 &= 9.8825 - 10 \\ \operatorname{colog} 647 &= 7.1891 - 10 \\ \log x &= 0.2174 \\ x &= [1.650]. \end{aligned}$$

It is a good custom to enclose a *computed* result in square brackets.

127. Arrangement of Work.—All logarithmic work should be arranged in a vertical column and should be done with pen and ink. Study the formula in which numerical values are to be substituted and decide upon an arrangement of your work in the vertical column which will make the additions, subtractions, etc. of logarithms as systematic and easy as possible. Fill out the vertical column with the names and values of the data before turning to the table of logarithms. This is called **blocking out the work.** The work is not properly blocked out unless every entry in the work as laid out is carefully labeled, stating exactly the name or the value of the magnitude whose logarithm is taken, and unless the computation sheet bears a formula or statement fully explaining the purpose of the work.

Computation Sheet (form *M*7) is suitable for general logarithmic computation.

Illustration 1.—Find the weight in pounds of a circular disk of steel of radius 2.64 ft. and thickness 0.824 in., if the specific gravity of the steel be 7.86

FORMULA: Call r the radius in feet and t the thickness in inches. Take 62.48 as the weight of 1 cu. ft. of water. Then the weight in pounds w is given by

$$w = \pi r^2 \left(\frac{t}{12}\right) (62.48)\ (7.86).$$

Work:

$$\begin{aligned} \log \pi &= 0.4971 \\ 2 \log r &= 0.8432 \\ \log t &= 9.9159 - 10 \\ \operatorname{colog} 12 &= 8.9208 - 10 \\ \log 62.48 &= 1.7951 \\ \log 7.86 &= \underline{0.8954} \\ \log w &= 2.8675 \\ w &= [737.0 \text{ lb.}]. \end{aligned}$$

Illustration 2.—Compute the value of x if

$$x = \sqrt[3]{\frac{0.1673 \times 2.142}{3.871}}.$$

Work:

$$\begin{aligned} \log 0.1673 &= 9.2235 - 10 \\ \log 2.142 &= 0.3308 \\ \operatorname{colog} 3.871 &= \underline{9.4122 - 10} \\ \text{sum} &= 8.9665 - 10 \\ \log x &= 9.6555 - 10 \\ x &= [0.4524]. \end{aligned}$$

REMARK 1: In writing a decimal fraction without an integral part, *always* place a zero before the decimal point; thus 0.1673.

REMARK 2: Note the order of logarithmic work: First, write the formula; next, block out the work by writing down the first column, as in illustrations above; finally, hunt up logarithms from table and place in second column of work.

REMARK 3: After addition note that $22.8681 - 20$ is written as 2.8681.

REMARK 4: In dividing a logarithm like $8.9665 - 10$ by 3, first call the expression $28.9665 - 30$ and then divide by 3. If division by 5 had been required, the dividend would, of course, have been called $48.9665 - 50$.

Exercises

1. Compute by logarithms the value of the following: $2.56 \times 3.11 \times 421$; $7.04 \times 0.21 \times 0.0646$; $3215 \times 12.82 \div 864$.

2. Compute the following by logarithms: $81^3 \div 17^4$; $158\sqrt[4]{0.52}$; $(343/892)^3$; $\sqrt{1893}\,\sqrt{1912/446^2}$.

3. Compute the following by logarithms: $(2.7182)^{1.408}$; $(7.41)^{-5/4}$; $(8.31)^{0.27}$.

4. Solve the following equations: $5^x = 10$; $3^{x-1} = 4$; $\log_x 71 = 1.21$.

5. Find the amount of \$550 for 15 years at 5 per cent compound interest.

6. A corporation is to repay a loan of \$200,000 by twenty equal annual payments. How much will have to be paid each year if money be supposed to be worth 5 per cent?

Let x be the amount paid each year. As a debt of \$200,000 is owed *now*, the *present value* of the twenty equal payments of x dollars each must add up to the debt or \$200,000. The sum of x dollars to be paid n years hence has a present worth of only

$$\frac{x}{(1.05^n)}$$

if money be worth 5 per cent compound interest. The present value, then, of x dollars paid 1 year hence, x dollars paid 2 years hence, and so on, is

$$\frac{x}{1.05} + \frac{x}{(1.05)^2} + \frac{x}{(1.05)^3} + \cdots + \frac{x}{(1.05)^{20}}.$$

This is a geometrical progression which can be summed by the usual formula.

The result in this case is the value of an annuity payable at the end of each year for 20 years that a present payment of \$200,000 will purchase. Four-place tables will not give more than three-place accuracy in this and the following problem. To get four- or five-place accuracy, six-place tables would be required.

7. It is estimated that a certain power plant costing \$220,000 will become entirely worthless except for a scrap value of \$20,000 at the end of 20 years. What annual sum must be set aside to amount to the cost of replacement at the end of 20 years, if 5 per cent compound interest is realized on the money in the depreciation fund?

Let the annual amount set aside be x. In this case the twenty equal payments are to have a value of \$200,000, 20 *years hence*, while in the preceding problem the payments were to be worth \$200,000 *now*. In this case, therefore,

$$x(1.05)^{19} + x(1.05)^{18} + x(1.05)^{17} + \cdots + x(1.05)^2 + x(1.05) + x = \$200{,}000.$$

The geometrical progression is to be summed and the resulting equation solved for x.

8. The population of the United States in 1790 was 3,930,000 and in 1910 it was 93,400,000. What was the average rate per cent increase for each decade of this period, assuming that the population increased in geometrical progression with a uniform ratio for the entire period?

9. Find the surface and the volume of a sphere whose radius is 7.211.

10. Find the weight of a cone of altitude 9.64 in., the radius of the base being 5.35 in., if the cone be made of steel of specific gravity 7.93.

11. Find the weight of a sphere of cast iron 14.2 in. in diameter, if the specific gravity of the iron be 7.30.

12. In 24 hours of continuous pumping, a pump discharges 450 gal. per minute; by how much will it raise the level of water in a reservoir having a surface of 1 acre? (1 acre = 43,560 sq. ft.)

128. Trigonometric Computations.—Logarithms of the trigonometric functions are used for computing the numerical value of expressions containing trigonometric functions, and in the solution of triangles. Right triangles, previously solved by use of the natural functions, are often more readily solved by means of logarithms (see Sec. **52**). The tables of trigonometric functions contain adequate explanation of their use, so that detailed instructions need not be given in this place. Two new matters of great importance are met with in the use of the logarithms of the trigonometric functions that do not arise in the use of a table of logarithms of numbers, which, on that account require especial attention from the student:

1. In interpolating in a table of logarithms of trigonometric functions, *the corrections to the logarithms of all cofunctions must be subtracted and not added.* Failure to do this is the cause of most of the errors made by the beginner.

2. To secure proper relative accuracy in computation, the S and T functions must be used in interpolating for the sine and tangent of small angles.

In the following work, four-place tables of logarithms are supposed to be in the hands of the students.

Exercises

1. A lateral face of a right prism, whose base is a square 17.45 ft. on a side, is cut in a line parallel to the base by a plane making an angle of 27° 15′ with the face. Find the area of the section of the prism made by the cutting plane.

2. The perimeter of a regular decagon is 24 ft. Find the area of the decagon.

3. To find the distance between two points B and C on opposite banks of a river, a distance CA is measured 300 ft., perpendicular to CB. At A the angle CAB is found to be 47° 27′. Find the distance CB.

4. In running a line 18 miles in a direction north, 2° 13.2′ east, how far in feet does one depart from a north-and-south line passing through the place of beginning?

5. How far is Madison, Wis., latitude 43° 5′, from the earth's axis of rotation, assuming that the earth is a sphere of radius 3960 miles?

6. A man walking east 7° 15′ north along a river notices that after passing opposite a tree across the river he walks 107 paces before he is in line with the shadow of the tree. Time of day, noon. How far is it across the river?

7. Solve the right-angled triangle in which one leg $= 2\sqrt{3}$ and the hypotenuse $= 2\pi$.

8. The moon's radius is 1081 miles. When nearest the earth, the moon's apparent diameter (the angle subtended by the moon's disk as seen from the position of the earth's center) is 32.79′. When farthest from the earth, her apparent diameter is only 28.73′. Find the nearest and the farthest distances of the moon in miles.

9. A pendulum 39 in. long vibrates 3° 5′ each side of its mean position. At the end of each swing, how far is the pendulum bob above its lowest position?

10. If the deviation of the compass be 2° 1.14′ east, how many feet does magnetic north depart from true north in a distance of 1 mile true north?

11. Solve

$$x : 1.72 = 427 : \sqrt{2gh},$$

if $g = 32.2$ and $h = 78.2$.

12. A substance containing 20 per cent of impurities is to be purified by crystallization from a mother liquid. Each crystallization reduces the impurity 88.6 per cent. How many crystallizations will produce a substance 0.9999 pure?

13. Compute the value of $(1 - ae^{-by})^n$ where $a = 15.6$, $b = 2\pi/\lambda$, $\lambda = 10$, $n = 2$, $y = 2.5$.

14. Find the volume of a cone if the angle at the apex be 15° 38′ and the altitude 17.48 in.

15. The angle subtended by the sun's diameter as seen from the earth is 32.06′ Find the diameter of the sun in miles, if the distance from the earth to the sun be 92,800,000 miles.

16. Compute by logarithms four values of p from the equation $p = 32.2d^{1.408}$, for $d = 2, 3, 4, 5$.

17. Solve $3^x = 405$ for the value of x.

18. Compute:

$$\frac{23.07 \times 0.1354 \times \sqrt{234}}{13.54}.$$

What advantage is there in using the cologarithm of the denominator?

129. The Subtangent of the Exponential Curve.—The student is expected to construct the curves described in the following exercises by the method of Sec. **116.** The inch or 2 cm. may be adopted as the unit of measure; the curves should be drawn on plain paper within the interval from $x = -2$ to $x = +2$.

If tangents[1] be drawn to the curves at $x = -2, -1, 0, 1, 2$, it will be noted, as nearly as can be determined by experiment, that the several tangents to any one curve cut the X-axis at the

[1] It is not easy to draw accurately the tangent to a curve at a given point. A number of instruments have been designed to assist in drawing tangents to curves. One of these, called a "Radiator," will be found listed in most catalogs of drawing instruments. Another instrument consists of a straight edge provided with a vertical mirror as shown in Fig. 95. When the straight edge is placed across

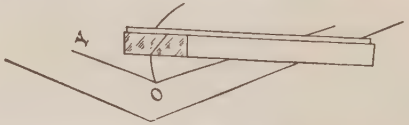


FIG. 95.—Mirrored ruler for drawing the normal (and hence the tangent) to any curve.

a curve, the reflection of the curve in the mirror and the curve itself can both be seen, and usually the curve and the image meet to form a cusp or angle. The straight edge may be turned, however, until the image forms a smooth continuation of the given curve. In this position the straight edge is perpendicular to the tangent and the tangent can then be accurately drawn. (See GRAMBERG, "Technische Messungen," 1911.)

same constant distance to the left of the ordinate of the point of tangency. This distance is called the **subtangent** of the curve. This distance is greater than unity if $a = 2$, and less than unity if $a = 3$. *The value of the base for which the subtangent is exactly unity is later shown to be a certain irrational or incommensurable number, approximately* 2.7183 . . . , *represented in mathematics by the letter e, and called the* **Naperian base.** This number, and the number π, are two of the most important and fundamental constants of mathematics.

Exercises

Draw the following curves on plain paper, using 1 in. or 2 cm. as the unit of measure; make the tests referred to in the second paragraph of Sec. **129.**

1. Construct a curve similar to Fig. 93, representing the equation $x = \log_2 y$, from $x = -2$ to $x = +2$, and draw tangents at $x = -1$, $x = 0$, $x = 1$, $x = 2$.

2. Construct the curve whose equation is $x = \log_3 y$ from $x = -2$ to $x = +2$, and draw tangents at $x = -1$, $x = 0$, $x = 1$, $x = 2$.

3. Construct the curve whose equation is $x = \log_{2.7} y$, and show by *trial* or experiment that the tangent to the curve at $x = 2$ cuts the X-axis at nearly $x = 1$, that the tangent at $x = 1$ cuts the X-axis at nearly $x = 0$, that the tangent at $x = 0$ cuts the X-axis at nearly $x = -1$, etc.

4. Draw the curve $x = \log_{0.5} y$ and show that it is the same as the reflection of $x = \log_2 y$ in the mirror $x = 0$.

NOTE: The student must remember that the experimental testing of the properties of the tangents to the curves called for above does not constitute *mathematical proof* of the usual deductive sort familiar to him. The experimental tests have value, however, in preparing the student for a rigorous investigation of these same properties when taken up in the calculus.

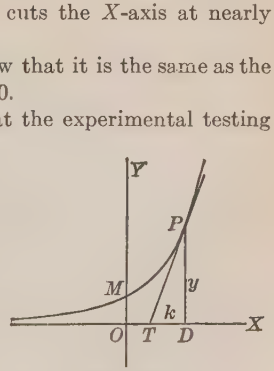


FIG. 96.

130.* Slope of the Exponential Curve.—Let MP (Fig. 96) be any exponential curve, $y = a^x$. By the **slope of the curve at P** we mean the slope of the tangent TP at P. We have just shown experimentally that the length of the subtangent TD is constant

for all positions of the point P on the curve $y = a^x$. We can then write

$$\text{slope of curve at } P = \frac{DP}{TD} = \frac{y}{k}, \tag{1}$$

where k is the constant length of TD. We can also write

$$\text{slope of curve at } P = cy, \tag{2}$$

where

$$c = \frac{1}{k}.$$

From Eq. (2) we can conclude that: *The slope of an exponential curve at a given point is proportional to the ordinate at that point.*

At the point (0, 1) the slope is $c = 1/k$. As we have seen, the value of k ($= TD$) depends on the value of a in the equation of the curve, $y = a^x$. For some values of a it is less than 1, for others greater than 1. We have defined e as that value of a for which $k = TD = 1$. This is equivalent to defining e as that value of a for which the curve $y = a^x$ has the slope unity at the point (0, 1).

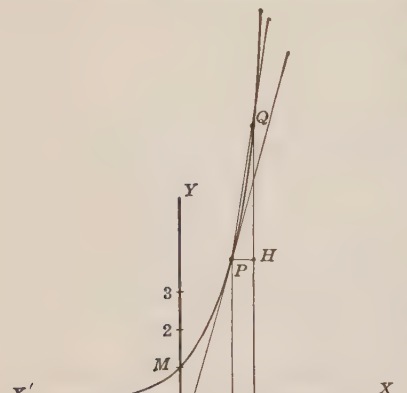


Fig. 97.—Definition of *tangent* to a curve.

When a has the special value e, k has the special value 1, and we conclude from Eq. (2) that *the slope of the exponential curve $y = e^x$ at a given point is numerically equal to the ordinate at that point.*

The two statements in italics made above are based upon the experimentally determined result that for a given exponential curve the subtangent is of constant length. It is of interest to see that these statements can be rigorously established. For this purpose it is only necessary to define the number e as that value of a for which the curve $y = a^x$ has unit slope at the point (0, 1).

Select any point P on the curve $y = e^x$. Draw a line through P cutting the curve at any neighboring point Q (Fig. 97). A line like PQ that cuts a curve at two points is called a **secant** line. As the point Q is taken nearer and nearer to the point P (P remaining fixed), the limiting position approached by the secant PQ is called the **tangent** to the curve at the point P. This is the general definition of the tangent to any curve.

The slope of the secant joining P to the neighboring point Q is HQ/PH. As the point Q approaches P this ratio approaches the slope of the tangent to $y = e^x$ at the point P. Let $OD = x$ and $PH = h$; then $OE = x + h$, also $DP = e^x$ and $EQ = e^{x+h}$. Since HQ is the y of the point Q minus the y of the point P, we have

$$\frac{HQ}{PH} = \frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h}.$$

Now the slope of $y = e^x$ at P is the limit of the above expression as Q approaches P, or as h approaches zero. That is,

$$\text{slope of } e^x \text{ at } P = \lim_{h \doteq 0} e^x \left[\frac{e^h - 1}{h}\right]. \tag{1}$$

As the point Q approaches the point P, or as h approaches zero, x does not change. Then

$$\text{slope of } e^x \text{ at } P = e^x \lim_{h \doteq 0} \left[\frac{e^h - 1}{h}\right]. \tag{2}$$

We now seek to find

$$\lim_{h \doteq 0} \left[\frac{e^h - 1}{h}\right],$$

if such limit exist.

Since the fraction $(e^h - 1)/h$ does not contain x, its value, for any value of h, and hence its limit, is the same for every point of the curve. If its value be calculated for any particular point of the curve, as the point M, it will have this value at any other point as P. From Eq. (2) the slope of the tangent line at the point M is

$$e^0 \lim_{h \doteq 0} \left[\frac{e^h - 1}{h}\right],$$

or

$$\lim_{h \doteq 0} \left[\frac{e^h - 1}{h}\right].$$

But by the definition of e, the slope of $y = e^x$ at M is **1.**

Hence,

$$\lim_{h \doteq 0} \left[\frac{e^h - 1}{h} \right] = 1. \tag{3}$$

Substituting this result in Eq. (2), we have

$$\text{slope at } P = e^x. \tag{4}$$

This expresses the fact that the slope of $y = e^x$ at any point is e^x, or is the ordinate y of that point.

The second statement made above is thus proved. It remains to prove the first. Now since there must be some number m such that $e^m = a$, the equation $y = a^x$ can be written $y = (e^m)^x = e^{mx}$, which shows that the curve $y = a^x$ is formed from the curve $y = e^x$ by multiplying all of the abscissas by a constant, $1/m$. In stretching the abscissas of the curve $y = e^x$ to produce the curve $y = a^x$ the base (such as TD in Fig. 97) of a triangle from which the slope of a tangent line can be computed will be multiplied by $1/m$, while the vertical leg (such as DP) will be unaltered. Hence the slope of the curve after stretching will be m times what it was originally; that is, the slope of $y = a^x$ at any point whose ordinate is y is m times the slope of $y = e^x$ at the point whose ordinate is y. But, since the slope of $y = e^x$ at any point is equal to the ordinate of that point, it follows that the slope of $y = a^x$ at any point is m times the ordinate of that point.

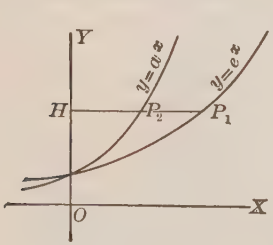


FIG. 98.—Comparison of $y = e^x$ and $y = a^x$.

131. Change of Base and Properties of the Exponential Curve.—Consider the curves $y = e^x$ and $y = a^x$ (Fig. 98), where $a > e$. For a given $y = OH$, the abscissas HP_1 and HP_2 are $\log_e y$ and $\log_a y$, respectively. It has been shown in the preceding section that the curve $y = a^x$ can be obtained from the curve $y = e^x$ by multiplying the abscissas of the latter curve by $1/m$, where m is the number such that

$$e^m = a. \tag{1}$$

In other words,

$$HP_2 = \frac{1}{m} HP_1.$$

That is,

$$\log_a y = \frac{1}{m} \log_e y. \tag{2}$$

As soon as m is known, we have a means of changing from a system of logarithms with base e to one with base a. *The number* $1/m$ *is called the* **modulus** *of the logarithmic system whose base is* a.

The **modulus of the common system** of logarithms is represented by M. It is the value of $1/m$ where m satisfies

$$e^m = 10, \text{ or } m = \log_e 10, \tag{3}$$

which is Eq. (1) for $a = 10$.
That is,

$$e^{1/M} = 10, \text{ or } e = 10^M.$$

Hence,

$$\mathbf{M = \log_{10} e = 0.4343.} \tag{4}$$

Hence, if N represents any number,

$$\mathbf{\log_{10} N = 0.4343 \log_e N,} \tag{5}$$

$$\mathbf{\log_e N = \frac{1}{0.4343} \log_{10} N = 2.3026 \log_{10} N.} \tag{6}$$

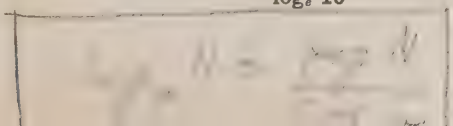
From the definition of m and M,

$$10^{0.4343} = e, \tag{7}$$

$$e^{2.3026} = 10. \tag{8}$$

Incidentally, it should be noted that, since $M = 1/m$, from Eqs. (3) and (4),

$$\mathbf{\log_{10} e = \frac{1}{\log_e 10}.} \tag{9}$$



A remarkable property of the logarithmic curve appears from comparing the curves $y = a^x$ and $y = a^{x+h}$. The second of these curves can be derived from $y = a^x$ by translating the latter curve the distance h to the left. But $y = a^{x+h}$ may be written $y = a^h a^x$, from which it may be seen that the new curve may also be considered as derived from $y = a^x$ by multiplying all ordinates of $y = a^x$ by a^h.

Translating the exponential curve in the negative x direction is the same as multiplying all ordinates by a certain fixed number, or is equivalent to a certain orthographic projection of the original curve upon a plane through the X-axis.

Changing the sign of h changes the sense of the translation and changes elongation to shortening, or *vice versa*.

Exercises

1. Compare the curve $y = e^x$ with the curve $y = 10^x$.

2. Graph the logarithmic spiral $\rho = e^\theta$, θ being measured in radians.

NOTE: The radian measure in the margin of form $M3$ should be used for this purpose.

3. Graph $\rho = e^{-\theta}$.

4. The pressure of the atmosphere is given in millimeters of mercury by the formula

$$y = 760 \cdot e^{-x/8000},$$

where the altitude x is measured in meters above the sea level. Produce a table of pressure for the altitudes $x = 0$; 10; 50; 100; 200; 300; 1000; 10,000; 100,000.

5. From the data of the last problem, find the approximate pressure at an altitude of 25,000 ft.

6. Show that the relation of Exercise 4 may be written

$$x = 18{,}421 \ (\log 760 - \log y).$$

7. Determining the value of the quotient $\ln x / \lg x$ for the following values of x: 2, 3, 5, 7.

8. How large is $e^{0.001}$, approximately?

9. What is the approximate value of $10^{0.001}$?

TABLE V

The following table of powers of e is useful in sketching exponential curves:

$e^{0.2} = 1.2214$	$e^{\frac{1}{2}} = 1.6487$	$e^{-0.2} = 0.8187$
$e^{0.4} = 1.4918$	$e^{\frac{1}{3}} = 1.3956$	$e^{-0.4} = 0.6703$
$e^{0.6} = 1.8221$	$e^{\frac{1}{4}} = 1.2840$	$e^{-0.6} = 0.5488$
$e^{0.8} = 2.2255$		$e^{-0.8} = 0.4493$
$e = 2.7183$		$e^{-1} = 0.3679$
$e^{2} = 7.3891$		$e^{-2} = 0.1353$
$e^{3} = 20.0855$		$e^{-3} = 0.0498$
$e^{4} = 54.5982$		$e^{-4} = 0.0183$

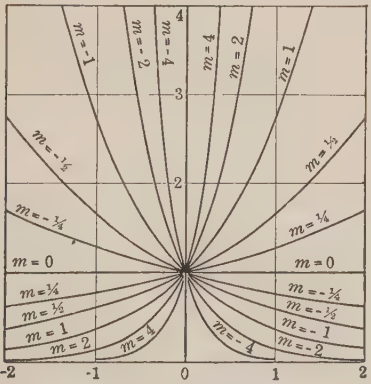


FIG. 99.—A family of exponentials, $y = e^{mx}$.

10. Draw the curve $y = e^{x} + e^{-x}$. Show that y is an *even* function of x, that is, that y does not change when the sign of x is changed.

11. Draw the curve $y = e^{x} - e^{-x}$. Show that y is an *odd* function of x, that is, that the function changes sign but not absolute value when the sign of x is changed.

12. Draw the graphs of $y = e^{2x}$, and $y = e^{-2x}$.

13. Draw the graphs of $y = e^{x/2}$, and $y = e^{-x/2}$.

14. Compare the curves: $y = e^{x/4}$, $y = e^{x/2}$, $y = e^{x}$, $y = e^{3x}$.

15. Sketch the curves $y = 1^{x}$, $y = 2^{x}$, $y = 3^{x}$, $y = 4^{x}$, $y = 5^{x}$, $y = 6^{x}$, $y = 8^{x}$, $y = 10^{x}$, from $x = -3$ to $x = +3$.

132. The Compound Interest Law. Logarithmic Decrement.—The law expressed by the exponential curve was called by Lord Kelvin the **compound interest law** and since that time this name has been generally used. It is recalled that the exponential curve

was drawn by using ordinates equal to the successive terms of a geometrical progression which are uniformly spaced along the X-axis. Since the amount of any sum at compound interest is given by a term of a geometrical progression, it is obvious that a sum at compound interest accumulates by the same law of growth as is indicated by a set of uniformly spaced ordinates of an exponential curve; hence the term "compound interest law" is appropriate. The compound interest law is one of the important laws of nature. As previously noted, the slope or rate of increase of the exponential function

$$y = ae^{bx}$$

at any point is always proportional to the ordinate or to the value of the function at that point. *Thus, when in nature we find any function or magnitude that increases at a rate proportional to itself* we have a case of the exponential or compound interest law. The law is also frequently expressed by saying, as has been repeatedly stated in this book, that *the first of two magnitudes varies in arithmetical progression while a second magnitude varies in geometrical progression.** A familiar example of this is the increased friction as a rope is coiled around a post. A few turns of the hawsers about the bitts at the wharf is sufficient to hold a large ship, because, as the number of turns increases in arithmetical progression, the friction increases in geometrical progression. Experiments to determine what weight could be held up by a 1-lb. weight, when a cord attached to the first weight passed over a round peg a number of times gave the following results:

n = number of turns of the cord on the peg	w = weight just held in equilibrium by 1-lb. weight	Logs of preceding numbers	d = logarithmic increment
½	1.6	0.204	.....
1	3.0	0.477	0.273
1½	5.1	0.708	0.231
2	8.0	0.903	0.195
2½	14.0	1.146	0.243
3	23.0	1.362	0.216
Average logarithmic increment =			0.23

*Compare the law of the power function, page 79, which law may be stated: "*In any power function of x, if x varies in geometrical progression, y will vary in geometrical progression also.*"

If the weights sustained were exactly in geometrical progression, their logarithms would be in arithmetical progression. The test for this fact is to note whether the differences between logarithms of successive values are constant. These differences are known as **logarithmic increments** or, in case they are negative, as **logarithmic decrements.** In the table the logarithmic increments fluctuate about the mean value 0.23.

133. Logarithmic Double Scales.—Figure 100 shows a logarithmic double scale (see Sec. **9**) so constructed that opposite any number y on the non-uniform scale is a number Y (on the uniform scale) which is equal to the logarithm of y, that is,

$$Y = \log y.$$

Actually to construct such a scale, first construct the uniform scale Y, in which the unit distance 0–1 is shown divided into 100 equal parts. Opposite 0.3010 (= log 2) of the Y-scale place a division line on the y-scale marked by the number 2. Opposite 0.4771 (= log 3) on the Y-scale place a division line on the y-scale marked 3; opposite 0.6021 (= log 4) on Y mark 4 on y, etc. Intermediate points on y are similarly located, for example, the 2.1 mark on y should be placed opposite 0.3222 (= log 2.1) on Y.

The double scale of Fig. 100 may obviously be used as a table of logarithms. Thus from it we may read log 7.1 = 0.85; log 3.3 = 0.52; log 1.5 = 0.175, etc.

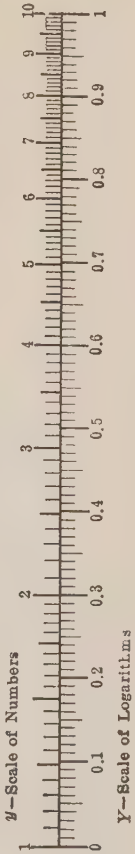


FIG. 100.—Logarithmic double scale.

Exercises

Read the following from the double scale (Fig. 100).

1. log 5.5. **3.** log 1.9. **5.** anti-log 0.74. **7.** anti-log 1.38.
2. log 2.4. **4.** log 71. **6.** anti-log 0.38. **8.** anti-log 2.38.

134. The Slide Rule.—By far the most important application of the non-uniform scale ruled proportionally to $\log x$ is the computing device known as the **slide rule.** The principle upon which the operation of the slide rule is based is very simple. If we have two scales divided proportionally to $\log x$ (A and B, Fig. 101), so arranged that one scale may slide along the other, then sliding one scale (called the slide) until its left end is opposite any desired division of the first scale, selecting any desired division of the slide, as at R (Fig. 101), and taking the reading of the original scale beneath this point, as N, the product of the two factors whose logarithms are proportional to AB and BR can be read directly from the lower scale at N. For AN is, by construction, the sum of AB and BR, and, since the scales were laid off proportionally to $\log x$ and marked with the numbers of which the distances are the logarithms, the process described adds the logarithms mechanically, but indicates the results in terms of the numbers themselves. By this device all of the operations commonly carried out by use of a logarithmic table may be performed mechanically. Full description of the use of the slide rule need not be given in detail at this place, as complete instructions are found in the pamphlet furnished with each slide rule. A very brief amount of individual instruction given to the student by the instructor will insure the rapid acquirement of skill in the use of the instrument. In what follows, the four scales of the slide rule are designated from top to bottom of the rule by A, B, C, D, respectively. The ends of the scales are called the **indices.**

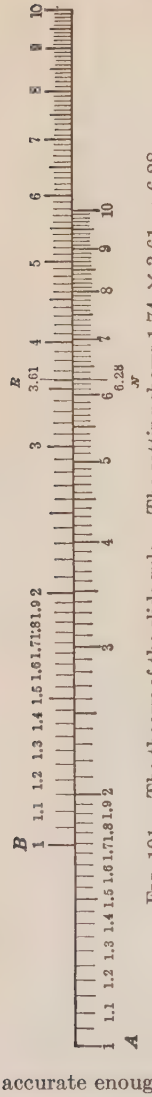


FIG. 101.—The theory of the slide rule. The setting shows $1.74 \times 3.61 = 6.28$.

An ordinary 10-in. slide rule should give results accurate to three significant figures, which is accurate enough for most of the purposes of applied science.

An exaggerated idea sometimes prevails concerning the degree of accuracy required by work in science or in applied science. Many of the fundamental constants of science, upon which a large number of other results depend, are known only to three decimal places. In such cases greater than three-figure accuracy is impossible, even if desired. In other cases greater accuracy is of no value, even if possible. The real desideratum in computed results is, first, *to know by a suitable check that the work of computation is correct*, and, second, *to know to what order or degree of accuracy both the data and the result are dependable.*

The absurdity of an undue number of decimal places in computation is illustrated by the original tables of logarithms, which if now used would enable one to compute from the radius of the earth the circumference *to one ten-thousandth part of an inch.*

The following matters should be emphasized in the use of the slide rule:

1. All numbers for the purpose of slide-rule computations should be considered as given with the first figure in units place. Thus, $517 \times 1910 \times 0.024$ should be considered as $5.17 \times 1.91 \times 2.4 \times 10^2 \times 10^3 \times 10^{-2}$. The result should then be mentally approximated (say, 24,000) for the purpose of locating the decimal point, and for checking the work.

2. A proportion should always be solved by one setting of the slide.

3. The runner must be set on the first half of A for square roots of numbers having an odd number of digits, and on the second half of A for the square roots of other numbers.

4. Use judgment so as to compute results in the most accurate manner—thus instead of computing 264/233, compute 31/233 and hence find $264/233 = 1 + 31/233$.[1]

5. Besides checking by mental calculation as suggested in Par. 1 above, also check by computing several neighboring values and graphing the results if necessary. Thus, check $5.17 \times 1.91 \times 2.4$ by computing both $5.20 \times 19.2 \times 2.42$ and $5.10 \times 1.90 \times 2.38$.

Exercises

Compute the following on the slide rule:

1. 3.12×2.24; 1.89×4.25; 2.88×3.16; 3.1×236.

2. 8.72/2.36; 4.58/2.36; 6.23/2.12; 10/3.14.

[1] Show by trial that this gives a more accurate result.

3. 32.5×72.5; 0.000116×0.00135; $0.0392/0.00114$.

4. $3{,}967{,}000 \div 367{,}800{,}000$.

5. $\dfrac{6.54 \times 42.6}{32.5}$; $\dfrac{8.75 \times 5.25}{32.3}$.

6. $\dfrac{78.5 \times 36.6 \times 20.8}{5.75 \times 29.5}$.

7. $\dfrac{6.46 \times 57.5 \times 8.55}{3.26 \times 296 \times 0.642}$.

8. Solve the proportion

$$x : 1.72 = 4.14 : \sqrt{2gh},$$

where $g = 32.2$ and $h = 78.2$.

9. Compute $\dfrac{\sqrt{171} \times 1.41}{166.7 \times 4.5}$.

10. The following is an approximate formula for the area of a segment of a circle:

$$A = \frac{h^3}{2c} + \frac{2ch}{3},$$

where c is the length of the chord and h is the altitude of the segment.

Test this formula for segments of a circle of unit radius, whose arcs are $\pi/3$, $\pi/2$, and π radians, respectively.

11. Two steamers start at the same time from the same port; the first sails at 12 miles an hour due south, and the second sails at 16 miles an hour due east. Find the bearing of the first steamer as seen from the second (*a*) after 1 hr., (*b*) after 2 hr., and compute their distances apart at each time.

The following exercises require the use of the data printed herewith. An "acre-foot" means the quantity of water that would cover 1 acre 1 ft. deep. "Second-foot" means a discharge at the rate of 1 cu. ft. of water per second. By the "run-off" of any drainage area is meant the quantity of water flowing therefrom in its surface stream or river, during a year or other interval of time.

1 square mile = 640 acres.
1 acre = 43,560 sq. ft.
1 day = 86,400 sec.
1 second-foot = 2 acre-ft. per day, approximately.
1 cubic foot = 7½ gal., approximately.
1 cubic foot water = 62½ lb. water, approximately.
1 horsepower = 550 ft.-lb. per second.
450 gallons per minute = 1 sec.-ft., approximately.

Each of the following problems should be handled on the slide rule as a continuous piece of computation.

12. A drainage area of 710 sq. miles has an annual run-off of 120,000 acre-ft. The average annual rainfall is 27 in. Find what per cent of the rainfall appears as run-off.

13. A centrifugal pump discharges 750 gal. per minute against a total lift of 28 ft. Find the theoretical horsepower required. Also daily discharge in acre-feet if the pump operates 14 hr. per day.

14. What is the theoretical horsepower represented by a stream discharging 550 sec.-ft. if there be a fall of 42 ft?

15. A district containing 25,000 acres of irrigable land is to be supplied with water by means of a canal. The average annual quantity of water required is $3\frac{1}{2}$ ft. on each acre. Find the capacity of the canal in second-feet, if the quantity of water required is to be delivered uniformly during an irrigation season of 5 months.

16. A municipal water supply amounts to 35,000,000 gal. per 24 hr. Find the equivalent in cubic feet per second.

17. A single rainfall of 3.9 in. on a catchment area of 210 sq. miles is found to contribute 17,500 acre-ft. of water to storage reservoir. The run-off is what per cent of the rainfall in this case?

135. Logarithmic Coordinate Paper.—Figure 102 represents a sheet of rectangular coordinate paper on which there are two scales drawn in each direction, namely, inner logarithmic scales (Sec. **133**), and outer uniform scales. Let us denote by x, y the coordinates of any point P referred to the non-uniform or logarithmic scales, and by X, Y the coordinates of P referred to the uniform scales. For the particular point P shown on the figure, for example, $x = 2$, $y = 2.2$, while $X = 0.30$, $Y = 0.35$. When the uniform scales are used, this is an ordinary sheet of Cartesian coordinate paper, and the equation of any straight line, such as PQ, is

$$Y = MX + B, \tag{1}$$

where M is the slope of the line ($\frac{1}{2}$ in the case of the line PQ) and B is the Y-intercept (0.2 in the case of the line PQ).

Now the two vertical scales, taken together, form a logarithmic double scale, so that (see Sec. **133**) for any pair of corresponding readings Y and y,

$$Y = \log y.$$

In particular,

$$B = \log b,$$

where b is the reading on the vertical non-uniform scale opposite B on the uniform scale. Similarly,

$$X = \log x.$$

Hence Eq. (1) may be rewritten

$$\log y = M \log x + \log b$$

or

$$\begin{aligned} \log y &= \log x^M + \log b \\ &= \log (bx^M), \\ y &= bx^M. \end{aligned} \tag{2}$$

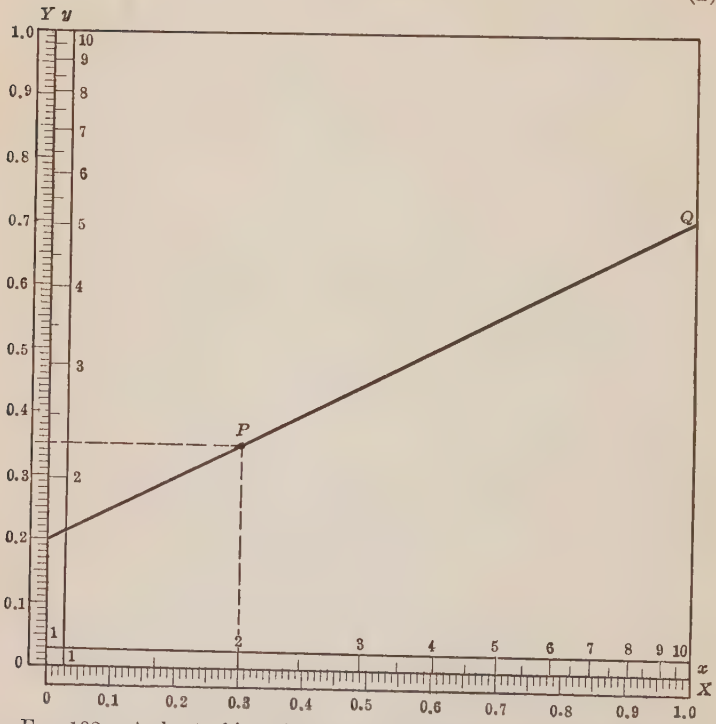


FIG. 102.—A sheet of logarithmic graph paper superposed on a sheet of Cartesian graph paper. No rulings are shown, since they would necessarily have to belong to one or the other system.

Thus, the straight line PQ, which is the graph of a linear function when referred to uniform scales, is the graph of the power function (Eq. (2)) when referred to the logarithmic scales. That is, *a power function graphs as a straight line when logarithmic scales are used.*

Referring to Eq. (2), the power M is the slope of the straight line PQ, while the coefficient b is the numerical reading at the point where the line PQ cuts the vertical non-uniform scale. Note especially that the scale on which b is to be read is not the "Y-axis" in the ordinary sense of that expression, but is the vertical line through $x = 1$.

Figure 103 shows a sheet of this same paper with rulings through the division points of the logarithmic scale, and with the uniform

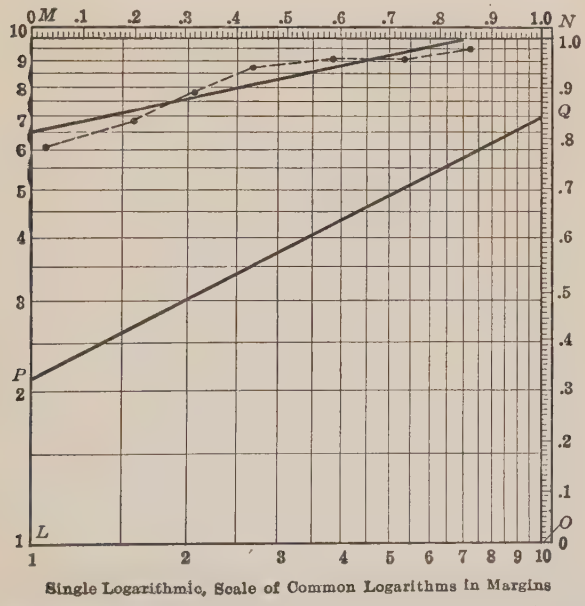


FIG. 103.—Logarithmic coördinate paper, form $M4$. The finer rulings of form $M4$ are not reproduced.

scales drawn on the right side and top of the square. Such squared paper is called **logarithmic paper.**

136. Graphing of Power Functions on Logarithmic Paper.— There are two principal uses of logarithmic paper. First, if one has a power function which he wishes to graph, this can be done

easily and accurately upon logarithmic paper, since the graph is a straight line. In connection with this use of logarithmic paper a few questions arise which can be best illustrated by examples.

In all of the following illustrations, it is understood that logarithmic paper is being used.

Illustration 1.—Construct the graph of the semi-cubical parabola $y = 2x^{3/2}$.

The result is a straight line of slope $3/2$ cutting the line LM (Fig. 103) at the point marked 2.

Illustration 2.—Graph the hyperbola $y = 6/x$.

The result is a straight line of slope -1 cutting the line LM (Fig. 103) at the point marked 6.

Illustration 3.—Graph the curve $y = 206x^{3.1}$.

Since the readings on the uniform scale ON are the logarithms of the corresponding readings on the scale LM, the numbers on LM may be multiplied by 10, provided 1 be added to the readings of ON; by 100, provided 2 be added to the readings of ON; etc. Thus, when LM reads 100, 200, 300, . . . 1000, the scale ON reads 2.0, 2.1, 2.2, . . . 3.0. The required graph is a straight line of slope 3.1, cutting the line LM at the point marked 206. Scale LM reads from 100 to 1000, scale LO reads from 1 to 10. Although, strictly speaking, scale ON now reads (as just explained) from 2.0 to 3.0, it is not necessary to renumber it, since neither in plotting points nor in reading off values is any use made of the uniform scales. They are drawn on the paper solely to enable one to compute the slope of the line conveniently.

Illustration 4.—Graph the curve $y = 0.086x^{-3}$.

The result is a straight line of slope -3 cutting the line LM at the point marked 0.086. Scale LM reads from 0.01 to 0.10, scale LO from 1 to 10.

137. Empirical Curves on Logarithmic Paper.—The second use of logarithmic paper referred to above, and the more important one, concerns the determination of the equation of a curve satisfied by laboratory data. If such data, when plotted on logarithmic paper, give a straight line, an equation of the form of Eq. (2) above satisfies the data, and the equation is readily found. The exponent M is determined by measuring the slope of the line with ordinary uniform scales. If the given data contain values of x ranging from 1 to 10, then the horizontal scale of the paper needs

no modification, and the value of b may be read by noting the point at which the line cuts the vertical non-uniform scale LM. If, however, the values of x range from, say, 20 to 85, the horizontal scale must be modified so as to read from 10 to 100. The vertical line on which b should be read (namely, the vertical line through $x = 1$) does not then appear on the paper. This difficulty may be avoided as follows: If x_0, y_0 are the coordinates of any point on the curve

$$y = bx^M,$$

then

$$y_0 = bx_0^M,$$

and, dividing the first equation by the second,

$$\frac{y}{y_0} = \left[\frac{x}{x_0}\right]^M. \tag{3}$$

That is, *if a straight line on logarithmic paper passes through the point* (x_0, y_0) *with slope* M, *its equation is given by* (3) *above.* Thus, b can be determined, once M is known, from the formula

$$b = \frac{y_0}{x_0^M}, \tag{4}$$

where x_0 and y_0 are the coordinates of any point on the line.

Illustration 1.—Find an empirical equation connecting the x and y of the following table:

x	1.40	2.10	3.20	4.50	6.30	8.00	9.20
y	2.46	3.04	3.70	4.48	5.23	6.01	6.40

These data require no modification of the original scales. When plotted, the points are found to lie nearly on the straight line PQ (Fig. 103) the discrepancies being such as might reasonably arise from inaccuracies in the observations. Assuming that, were the data exact, the points would lie exactly on this line,[1] the desired equation

[1] We would not be at liberty to make such an assumption if the variation of the points away from the line was of a character similar to that represented by the dots near the top of Fig. 103. These points, although not departing greatly from the line shown, depart from it systematically. That is, they lie below it at each end, and above it in the center, seeming to approximate a curve (such as the one shown dotted) more nearly than the line. The points arranged about the line PQ depart as far from that line as do the points above the higher line, but they do not depart systematically, as if tending to lie along a smooth curve. When points arrange themselves as at the top of Fig. 103, one must infer that the relation connecting the given data is not exponential in character.

is found at once by measuring the slope, which is found to be 0.50; and noting the point, 2.1, at which the line cuts the scale LM. The equation is hence $y = 2.1x^{0.5}$.

Illustration 2.—Find an empirical equation connecting the x and y of the following table:

x	110	240	325	490	630	835
y	6.11	3.83	3.19	2.50	2.16	1.83

To plot these data, it is necessary to modify the LO scale so as to read from 100 to 1000. The graph is found to be a straight line of slope -0.595. To determine b, choose any point on this line, such as the point ($x_0 = 490$, $y_0 = 2.5$). Then the desired equation may be written, from Eq. (3), as

$$\frac{y}{2.5} = \left[\frac{x}{490}\right]^{-0.595},$$

or

$$y = 99.68x^{-0.595}.$$

Note that 99.68 is *not* the point at which the line cuts the vertical scale at the left side of the paper.

Illustration 3.—Find an empirical equation connecting the x and the y of the following table:

x	y	x	y
5	1.0	20	16.4
7	2.0	30	37.0
9	3.2	40	65.0
15	9.3	50	100.0

The values of x in this table extend beyond any one decimal interval, such as 1–10, 10–100, etc. It is necessary either to use several sheets (in this case two) of ordinary logarithmic paper, placed side by side, or to use a sheet of "multiple logarithmic paper" on which is drawn, to a smaller scale, several squares of logarithmic ruling. Such a sheet of multiple logarithmic paper is shown in Fig. 104. The line PQ, plotted from the values in the above table, is found to have

slope 2.0. Substituting for x_0 and y_0 in Eq. (3) the coordinates of any of the points on the line, for example (5, 1), we get

$$\frac{y}{1} = \left(\frac{x}{5}\right)^2 \text{ or } y = \frac{x^2}{25}.$$

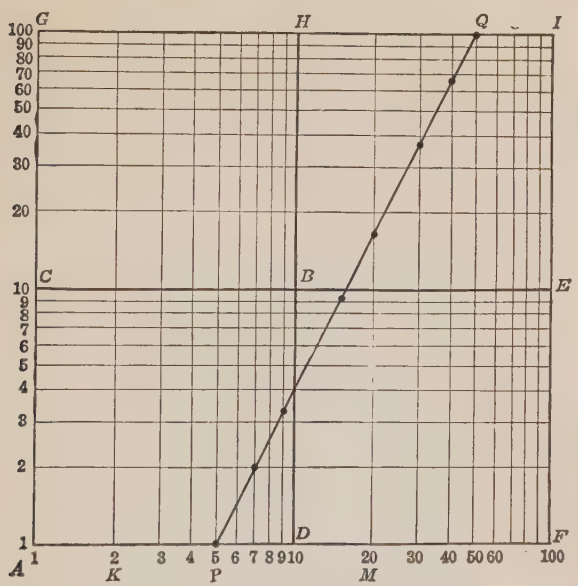


FIG. 104.—Multiple logarithmic paper.

Exercises

Draw the following on single or multiple logarithmic paper (forms *M*4 or *M*6):

1. $y = x$, $y = 2x$, $y = 3x$, $y = 4x$.
2. $y = x$, $y = x^2$, $y = x^3$, $y = x^4$.
3. $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$.
4. $y = x^{1/2}$, $y = x^{1/3}$, $y = x^{2/3}$.
5. $y = 2x^3$, $y = \frac{1}{3}x^2$, $A = \pi r^2$.
6. $y = 71.3x^{2.5}$.
7. $y = 420x^{-0.7}$.
8. $y = 0.037x^{2.1}$.
9. $y = 0.00017x^{-2.17}$.
10. $y = 2x^{10.1}$.

NOTE: If the line of Exercise 10 be drawn, with slope 10.1, through the point 2 on the *LM* scale of form *M*4, the horizontal range of values

covered extends only from 1 to approximately 1.2. This range can be increased by drawing a second line, parallel to the first, through that point of the LO scale which is directly under the intersection of the original line and the MN scale. This amounts, as the student should convince himself, to using a second square of paper, located above the first. When reading values off this second line, what modification must be made in scale LM?

11. Draw on one sheet of form $M4$, a graph of the function $y = 4x^3$ which can be used for any values of x between 1 and 10.

12. Find the empirical equation connecting x and y of the following table:

x	y	x	y
1.5	3.05	6.5	6.40
2.5	3.92	7.5	6.85
3.5	4.65	8.5	7.25
4.5	5.30	9.5	7.70
5.5	5.82	.....	.....

13. Find the empirical equation connecting x and y of the following table:

x	y	x	y
1.2	2.15	2.0	5.90
1.3	2.50	2.3	7.80
1.5	3.35	2.5	9.30
1.7	4.30	.....	.....

14. Find the empirical equation connecting x and y of the following table:

x	y	x	y
1.5	10.0	4.5	3.30
2.0	7.5	5.0	2.98
2.5	6.0	6.0	2.49
3.0	5.0	7.0	2.12
3.5	4.25	8.0	1.87
4.0	3.73	9.0	1.65

15. Find the empirical equation connecting x and y of the following table:

x	y	x	y
11	176	39	290
19	220	44	305
25	243	53	329
31	265	70	367

16. Find the empirical equation connecting x and y of the following table:

x	y	x	y
1.6	0.0093	3.7	0.0036
2.1	0.0069	4.3	0.0031
2.9	0.0048	5.1	0.0025
3.3	0.0041	7.2	0.0017

17. Find the empirical equation connecting x and y of the following table:

x	y	x	y
1.1	1.22	21.0	2.60
3.1	1.60	35.0	3.00
6.9	1.97	64.0	3.45
12.0	2.28	80.0	3.70

Draw the following on single or multiple logarithmic paper as best suits the particular example. Carefully label the scales and indicate the true numerical value of the division points. Use common-sense values of the variables—for example, in Exercise 26 do not graph for speed over 30 knots.

18. $p = 0.003v^2$, where p is the pressure in pounds per square foot on a flat surface exposed to a wind velocity of v miles per hour.

SUGGESTION: The common-sense range for v is from $v = 10$ to $v = 100$.

19. Find the equations of the lines EF and GH of Fig. 105.

20. $v = c\sqrt{rs}$ for $c = 110$ and $r = 1$.

21. $f = \sqrt{2gh}$ for $g = 32.2$.

22. $C = E/R$ where $E = 110$ volts.

23. $s = \frac{1}{2}gt^2$ where $g = 32.2$.

24. $T = \pi\sqrt{L/g}$ where $g = 32.2$.

25. $p/p_0 = (\rho/\rho_0)^{1.408}$ where $\rho_0 = 0.075$, the weight of 1 cu. ft. of air in pounds at 70°F. and at pressure p_0 of 14.7 lb. per square inch.

26. $H = S^3D^{\frac{2}{3}}/C$ for $D = 5000, 10{,}000, 15{,}000, 20{,}000$, where $C = 225$, D is displacement in tons and S is speed in knots.

27. $q = 3.37LH^{3\frac{1}{2}}$ for $L = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ (see Fig. 106).

Note that in this figure H is vertical and q horizontal. This explains why the lines have the slope $\frac{2}{3}$, rather than $3\frac{1}{2}$, as would be the case if q were vertical.

138. Semi-logarithmic Paper.—Figure 107 represents a sheet of rectangular coordinate paper on which there is a uniform horizontal scale, and two vertical scales, namely, a uniform scale and a logarithmic scale. Let us denote by y the ordinate of any point P referred to the non-uniform or logarithmic scale and by Y the ordinate of P referred to the uniform scale. The abscissa of P, as read on the horizontal uniform scale, will be denoted by x. Now when the uniform vertical scale Y and the uniform horizontal scale x are used, this is an ordinary sheet of Cartesian graph paper, and the equation of any straight line, such as PQ, is

$$Y = Mx + B, \tag{1}$$

where M is the slope of the line ($-\frac{1}{3}$ for the line PQ) and B is the Y-intercept (0.7 for the line PQ).

The two vertical scales, taken together, form a logarithmic double scale, so that (see Sec. **133**), for any pair of corresponding readings Y and y,

$$Y = \log y.$$

In particular,

$$B = \log b,$$

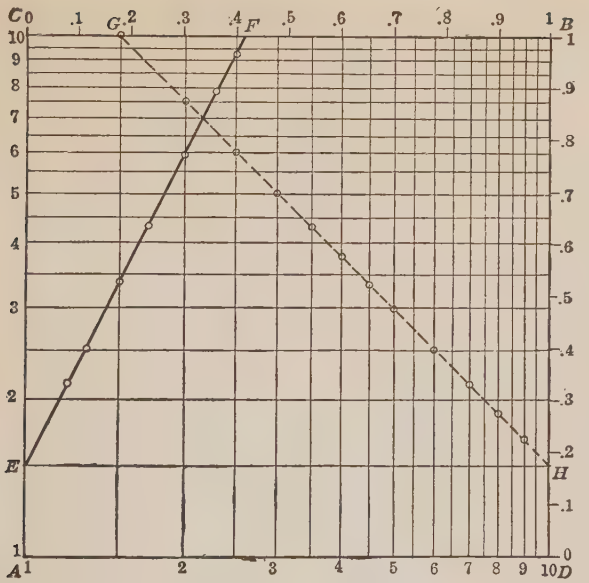


FIG. 105.—Diagram for Exercise 19.

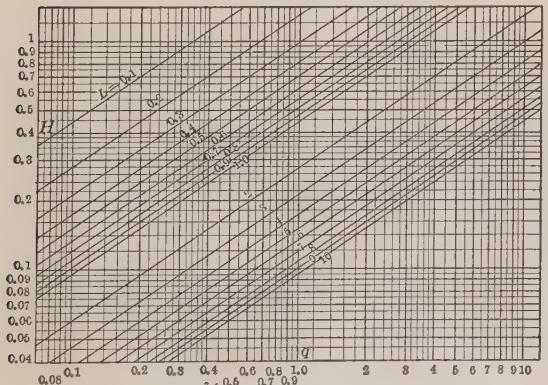

H
q
Formula: $q = 3.37 L H^{3/2}$
Discharge over Trapezoidal Wier
L = Length of Crest in Feet
H = Head on Crest in Feet
q = Discharge in Second Feet

FIG. 106.—A weir formula graphed on multiple logarithmic paper.

where b is the reading on the vertical non-uniform scale opposite B on the uniform scale. Substituting these values of Y and B in Eq. (1),

$$\log y = Mx + \log b,$$

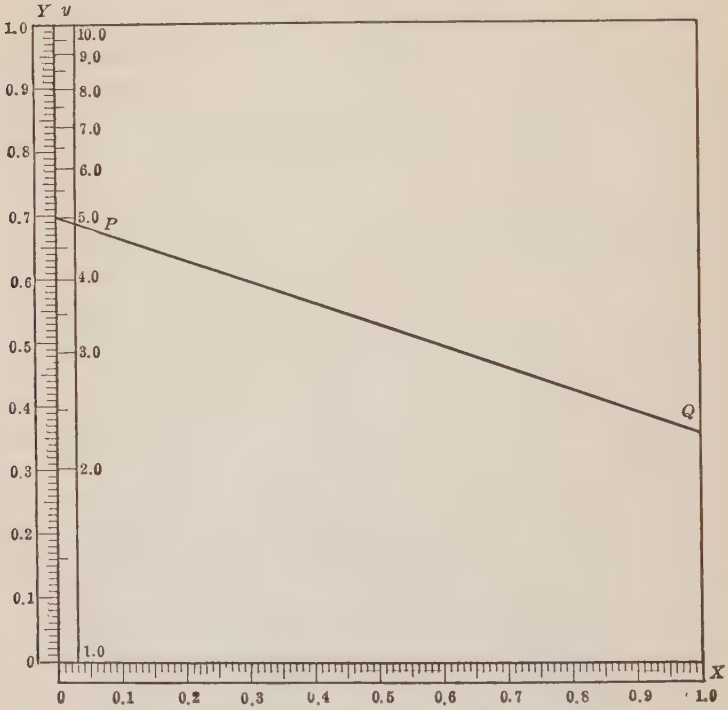


FIG. 107.—The theory of the use of semi-logarithmic paper.

so that

$$\log\left(\frac{y}{b}\right) = Mx$$

or

$$\mathbf{y = b10^{Mx}}. \tag{2}$$

Thus, the straight line PQ, which is the graph of a linear function when referred to uniform scales, is the graph of the

exponential function (Eq. (2)) when referred to a uniform horizontal scale and a logarithmic vertical scale. That is, *an exponential function graphs as a straight line when a uniform horizontal and a vertical logarithmic scale are used.* Referring to Eq. (1), M is the slope of the straight line PQ, while b is the numerical reading at the point where the line PQ cuts the vertical non-uniform scale. Note that the line on which b is to be read is the Y-axis, that is to say, is the vertical line through $x = 0$.

Figure 108 shows a sheet of this same paper with vertical rulings through the division points of the horizontal scale, and horizontal rulings through the division points of the vertical logarithmic scale. The uniform horizontal scale is drawn and numbered at both the bottom and the top of the sheet.[1] The vertical uniform scale is drawn at the right-hand side of the sheet, and is not numbered. The uniform unnumbered scale AD at the left side of the sheet is for a purpose not mentioned as yet, and will be explained later. Such squared paper as that shown in Fig. 108 is called **semi-logarithmic paper.** As was the case with logarithmic paper, semi-logarithmic paper can be used either to graph an exponential equation, or to determine an empirical equation satisfied by experimental data.

Illustration 1.—Draw the curve $y = 2(10^{x/3})$ on semi-logarithmic paper.

The result is a straight line of slope $\frac{1}{3}$ cutting the logarithmic scale LM (Fig. 108), at the point marked 2.

Illustration 2.—Draw the curve $y = 35(10^{-0.2x})$.

The given value of b requires the vertical scale LM to be modified so as to read from 10 to 100. Scale ON then reads from 1.0 to 2.0, although, as before, there is no necessity for numbering it, since it is used only in determining the slope. Scale AB is unmodified, and reads from 0.0 to 1.0. The result is a straight line of slope -0.2 cutting the scale LM at the point marked 35.

Illustration 3.—Draw the curve $y = 0.086\ 10^{3x}$.

The result is a straight line of slope 3 cutting the scale LM at the point numbered 0.086. Scale LM reads from 0.01 to 0.10, scale AB from 0.0 to 1.0.

[1] The reversed numbering of the upper horizontal scale is often useful in measuring the slope of a line which cuts it, but which does not cut the lower horizontal scale

Illustration 4.—Draw the curve $y = 2.7e^{0.8x}$.

Since $10^{0.4343} = e$, this equation can be written $y/2.7 = 10^{(0.4343)(0.8)x}$. Therefore the graph is a straight line cutting the scale LM at the point 2.7, and of slope $(0.4343)(0.8)$. To lay off conveniently a line of this slope, the auxiliary uniform scale AD is provided. The scale AD is a uniform scale obtained from ON by shortening all lengths in the ratio of 0.4343 to 1. Unity on this scale, hence, is opposite the point 0.4343 on ON, and the point 2.718 on LM. To lay off a slope of $(0.4343)(0.8)$, one proceeds as if he were laying off

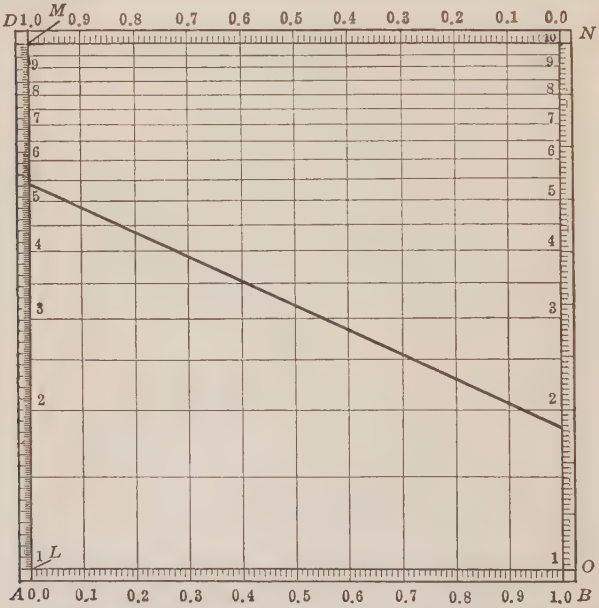


FIG. 108.—Illustration of squared paper, form $M5$. The finer rulings of form $M5$ have, however, been omitted.

a slope of 0.8, but uses the scale AD for measuring vertical distances, which process automatically introduces the factor 0.4343.

Illustration 5.—Find an empirical equation connecting the x and the y of the following table:

x	0.040	0.180	0.360	0.510	0.685	0.833	0.970
y	5.300	4.400	3.750	3.100	2.600	2.330	1.900

These data require no modification of the original scales. The resulting straight line is shown in Fig. 108. Its slope is found to be -0.447, while it cuts the scale LM at the point 5.39. The desired equation is, therefore, $y = 5.39(10^{-0.447x})$. Since $10 = e^{2.3026}$ (see Sec. **131**) this can also be written in the form $y = 5.39(e^{2.306})^{-0.447x} = 5.39e^{-1.029x}$. The same result could have been obtained directly by computing the slope of the line from scales AD and LO, rather than from ON and LO.

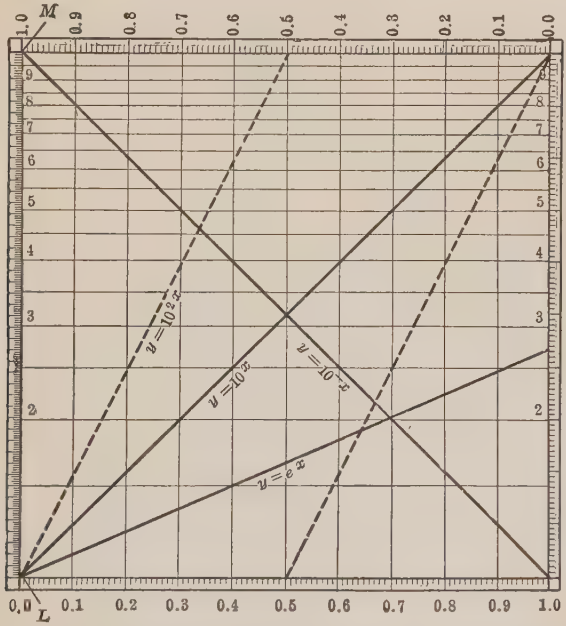


FIG. 109.—Semi-logarithmic coordinate paper.

Illustration 6.—Find an empirical equation connecting the x and the y of the following data:

x	0.86	2.10	5.00	6.30	7.82	9.10
y	2.23	2.68	4.00	4.76	5.90	7.05

To plot these data it is necessary to modify the scale AB so as to read from 0 to 10. Scale LM reads, as before, from 1 to 10 and hence scale ON reads, as before, from 0 to 1. The straight line which these

data locate cuts LM at the point 2.0. In determining its slope one must remember that the slope is measured relative to the uniform scales AB and ON. It is found, on these scales, that in going a horizontal distance of 10 units (that is, from A to B) the straight line rises, as measured on ON, 0.6 units. The slope is, hence, $0.6/10 = 0.06$. The desired equation is, hence, $y = 2.0(10^{0.06x})$. If it is desired to use e as a base, the slope should be measured by means of AD and AB. In going a horizontal distance of 10 units, the line rises, as measured on AD, 1.38 units, so that its slope, as measured on these scales, is 0.138. The desired equation is, hence, $y = 2e^{0.138x}$.

Exercises

On semi-logarithmic paper draw the following:

1. $y = 10^{3x}$, $y = 10^{2x}$, $y = 10^{x}$.

2. $y = 10^{-x}$, $y = 10^{-2x}$, $y = 10^{-3x}$.

3. $y = e^{2x}$, $y = e^{x}$.

4. $y = e^{-x}$, $y = e^{-2x}$.

5. $3x = \log y$, $(\frac{1}{2})x = \log y$.

6. $y = 10^{x/2}$, $y = 10^{x/10}$.

7. Graph $y = 2(10)^{x}$ and $y/2 = 10^{x-3}$.

8. Referring to Fig. 109, how does scale LM read when one reads values off the line $y = 10^{-x}$?

9. Referring to Fig. 109, explain the relation of the unlabeled dotted line to the dotted line labeled $y = 10^{2x}$. When using the unlabeled line, how does scale LM read?

10. A thermometer bulb initially at temperature 19.3°C. is exposed to the air and its temperature θ noted at various times t (in seconds) as follows:

t	0	20	40	60	80	100	120
θ	19.3	14.2	10.4	7.6	5.6	4.1	3.0

Plot these results on semi-logarithmic paper and test whether or not θ follows the compound interest law. If so, determine the value of θ_0 and b in the equation $\theta = \theta_0 10^{-bt}$. Note that the last point given by the table, namely, $t = 120$, $\theta = 3.0$, goes into a new square if the scale AB be called 0-100. If the scale AB be called 0-200, then all entries can appear on a single sheet of form $M5$.

11. Graph the following on semi-logarithmic paper:

n	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
w	1.6	3.0	5.1	8.0	14.0	23.0

Show that the equation connecting n and w is $w = 10^{0.45n}$. The data of this problem are taken from Sec. **132.**

SUGGESTION: Scale AB (form $M5$), may be called 0-5 for the purpose of graphing n. Since the values of w are not contained in one decimal interval, it is necessary to use two sheets of the paper, one above the other.

12. Graph the following on semi-logarithmic paper, and find the equation connecting n and w:

n	0.2	0.4	0.6	0.8	1.0	1.2
w	2.60	3.41	4.45	5.75	7.56	9.85

13. A circular disk is suspended in a horizontal plane by a fine wire at its center. When at rest, the upper end of the wire is turned by means of a supporting knob through 30°. The successive angles of the torsional swings of the disk from the neutral point are then read at the end of each swing as follows:

Swing number	1	2	3	4	5	6	7
Angle	26.4°	23.2°	20.5°	18.0°	15.9°	14.0°	12.3°

Show that the angle of the successive swings follows the compound interest law and find in at least two different ways, the equation connecting the number of the swing and the angle. Show also by the slide rule that the compound interest law holds. [$\phi = 30 \cdot 10^{-.055n}$.]

14. Draw the curve $y = 37.3(10^{0.8x})$.

15. Draw the curve $y = 0.0071(10^{2x})$.

16. Draw the curve $y = 8.2e^{3.1x}$.

17. Find an empirical equation connecting the x and the y of the following table:

x	y	x	y
3.08	170	3.53	500
3.21	232	3.64	650
3.30	288	3.79	935
3.41	372	3.82	1000

SUGGESTION: Note that AB should read from 3 to 4, and LM from 100 to 1000. The vertical line on which b can be read, namely, the

vertical line through $x = 0$, does not appear on the paper. This difficulty can be met as was done with logarithmic paper. If x_0, y_0 be any point on the line, then

$$\frac{y}{y_0} = \frac{10^{mx}}{10^{mx_0}} = 10^{m(x-x_0)} = 10^{-mx_0}\,10^{mx},$$

so that

$$b = y_0\,10^{-mx_0}.$$

18. Find an empirical equation connecting the x and the y of the following table:

x	y	x	y
3.30	.00650	5.70	.00305
3.95	.00530	6.35	.00250
4.45	.00450	7.05	.00200
5.00	.00380	7.85	.00155

SUGGESTION: Scale AB should read from 3–8, scale LM, 0.001–0.01.

139. Sums of Exponential Functions.—Functions consisting of the sum of two different exponential functions are of frequent occurrence in the application of mathematics, especially in electrical science. Types of fundamental importance are $e^u + e^{-u}$ and $e^u - e^{-u}$, which are so important that the forms $(e^u + e^{-u})/2$ and $(e^u - e^{-u})/2$ have been given special names and tables of their values have been computed and printed. The first of these is called the **hyperbolic cosine of u** and the second is called the **hyperbolic sine of u**; they are written in the following notation:

$$\cosh u = \frac{(e^u + e^{-u})}{2}, \quad \sinh u = \frac{(e^u - e^{-u})}{2}.$$

If $x = a \cosh u$ and $y = a \sinh u$, then, squaring and subtracting,

$$x^2 - y^2 = a^2(\cosh^2 u - \sinh^2 u)$$
$$= a^2\left[\frac{e^{2u} + 2 + e^{-2u}}{4} - \frac{e^{2u} - 2 + e^{-2u}}{4}\right] = a^2.$$

Therefore the hyperbolic functions

$$x = a \cosh u, \text{ and } y = a \sinh u$$

appear in the parametric equations of a rectangular hyperbola

$$x^2 - y^2 = a^2,$$

just as the circular functions

$$x = a \cos \phi, \text{ and } y = a \sin \phi$$

appear in the parametric equations of the circle

$$x^2 + y^2 = a^2.$$

The graphs of $y = a \cosh x$ and $y = a \sinh x$ were called for in Exercises 10 and 11, (Sec. **131**). They are shown in Fig. 110.

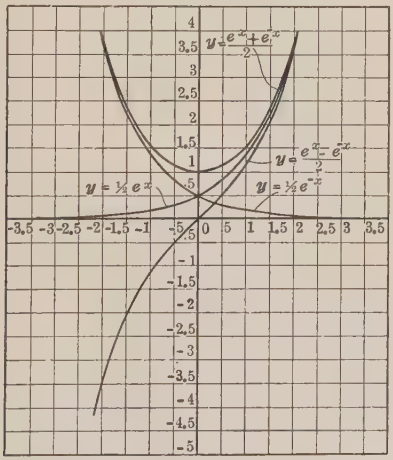


FIG. 110.—The curves of the hyperbolic sine and cosine.

The first of these curves is formed when a chain is suspended between two points of support; it is called the **catenary.** These two curves are best drawn by averaging the ordinates of $y = e^x$ and $y = e^{-x}$, and the ordinates of $y = e^x$ and $y = -e^{-x}$.

Curves whose equations are of the form

$$y = ae^{mx} + be^{nx}$$

take on a variety of forms for various values of the constants. A good idea of certain important types can be had by a comparison of the curves of Figs. 111 and 112.

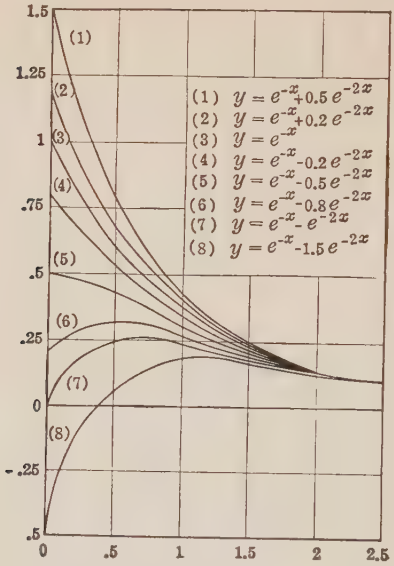


FIG. 111.

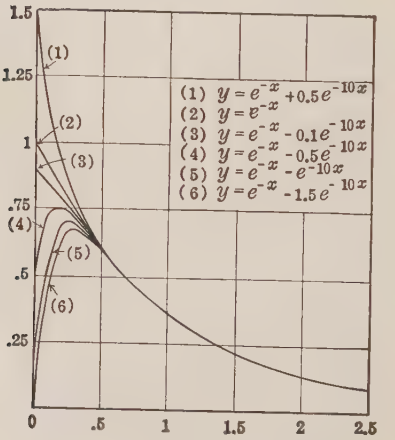


FIG. 112.

The student should arrange in tabular form the necessary numerical work for the construction of the curves of Figs. 111 and 112.

If the coefficient of the second exponent be increased in absolute value, the points of intersection with the Y-axis remain the same, but the region of close approach of the curves to each other is moved along the curve $y = e^{-x}$ to a point much nearer the Y-axis, as may be seen by comparing Fig. 112 with Fig. 111.

140.* Damped Vibrations.—If a body vibrates in a medium like a gas or a liquid, the amplitude of the swings are found to get smaller and smaller, or the motion slowly (or rapidly in some cases) dies out. When a pendulum vibrates in oil, the rate of decay of the amplitude of the swings is rapid, but the ordinary rate of the decay of such vibrations in air is slow. The ratio between the lengths of the successive amplitudes of vibration is called the **damping factor** or the **modulus of decay.**

The same fact is noted in case the vibrations are the torsional vibrations of a body suspended by a fine wire or thread. Thus, a **viscometer,** an instrument used for determining the viscosity of lubricating oils, provides means for determining the rate of the decay of the torsional vibration of a disk, or of a circular cylinder suspended in the oil by a fine wire. The "amplitude of swing" is in this case the angle through which the disk or cylinder turns, measured from its neutral position to the end of each swing.

In all such cases it is found that the logarithms of the successive amplitudes of the swings *differ by a certain constant amount* or, as it is said, the *logarithmic decrement is constant.* Therefore the amplitudes must satisfy an equation of the form

$$A = ae^{-bt}$$

where A is amplitude and t is time. The actual motion is given by an equation of the form

$$y = ae^{-bt} \sin ct.$$

A study of oscillations of this type will be taken up more fully in the calculus. For the present it will suffice to graph a few examples. Let the expression be

$$y = e^{-t/5} \sin t. \tag{1}$$

A table of values of t and y must first be derived. There are three ways of proceeding: (1) Assign successive values to t irrespective of the period of the sine. (2) Select for the values of t those values that

give aliquot parts of the period 2π of the sine (see Table VI and Fig. 113). (3) Draw the sinusoid $y = \sin t$ carefully to scale by the method of Sec. **58**; then draw upon the same coordinate axes, using the same units of measure adopted for the sinusoid, the exponential curve $y = e^{-t/5}$; finally, multiply together, on the slide rule, corresponding ordinates taken from the two curves, and locate the points thus determined

The first method involves very much more work than the second for two principal reasons: First, tables of the logarithms of the

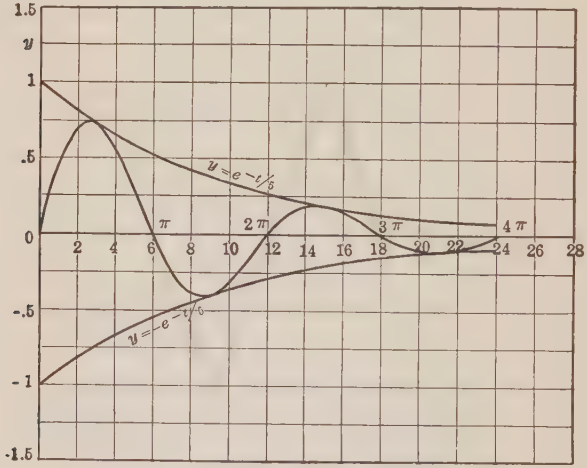


FIG. 113.—The curve $y = e^{-t/5} \sin t$.

trigonometric functions with the radian and the decimal divisions of the radian as argument are not available; for this reason 57.3° must be multiplied by the value of t in each case so that an ordinary trigonometric table may be used; second, each of the values of log sin t must be separately determined, while, if the periodic character of the sine be taken advantage of, the numerical values would be the same in each quadrant.

The second method, because of the use of aliquot divisions of the period of the sine, such as $\pi/6$ or $\pi/12$ or $\pi/18$ or $\pi/20$, etc., possesses the advantage that the values used in column 3 need be found for one

quadrant only and the values required in column 2 are quite as readily found on the slide rules as in the first method.

TABLE VI.—TABLE OF THE FUNCTION $y = e^{-t/5} \sin t$

1	2	3	4	5
$n = t$ in units of $\pi/6$ radians	$\log e^{-n\pi/30} = -(0.0455)n$	$\log \sin n\pi/6$	$\log y$	y
0	− 0.0000	.....	.....	0.000
1	− 0.0455	9.6990	9.6535	+ 0.450
2	− 0.0910	9.9375	9.8465	+ 0.702
3	− 0.1364	0.0000	9.8636	+ 0.731
4	− 0.1819	9.9375	9.7556	+ 0.570
5	− 0.2274	9.6990	9.4716	+ 0.296
6	− 0.2729	.....	.....	+ 0.000
7	− 0.3184	9.6990	9.3806	− 0.240
8	− 0.3638	9.9375	9.5737	− 0.375
9	− 0.4093	0.0000	9.5907	− 0.390
10	− 0.4548	9.9375	9.4827	− 0.304
11	− 0.5003	9.6990	9.1987	− 0.158
12	− 0.5458	.....	.....	0.000
13	− 0.5912	9.6990	9.1078	+ 0.128
14	− 0.6367	9.9375	9.3008	+ 0.200
15	− 0.6822	0.0000	9.3178	+ 0.208
16	− 0.7277	9.9375	9.2098	+ 0.162
17	− 0.7732	9.6990	8.9258	+ 0.084
18	− 0.8186	.....	.....	0.000
19	− 0.8641	9.6990	8.8349	− 0.068
20	− 0.9016	9.9375	9.0279	− 0.107
21	− 0.9551	0.0000	9.0449	− 0.111
22	− 1.0006	9.9375	8.9369	− 0.087
23	− 1.0460	9.6990	8.6530	− 0.045
24	− 1.0915	.....	.....	0.000

The third method is perhaps more desirable than either of the others if greater than two-figure accuracy is not required. The curve can readily be drawn with the scale units the same in both dimensions, as is sometimes highly desirable in scientific applications.

In Fig. 113 a larger unit has been used on the vertical scale than on the horizontal scale, and the horizontal unit is incommensurable with the vertical unit. To draw the curve to a true scale in both dimensions it is preferable to lay off the coordinates on plain drawing

paper and not on ordinary squared paper. Rectangular coordinate paper is not adapted to the proper construction and discussion of the sinusoid, or of curves, like the present one, that are derived therefrom.

Curves whose equations are of the form $y = \frac{1}{2}e^{-t/5} \sin t$ or $y = 3e^{-t/5} \sin t$, etc. are readily constructed, since the constants $\frac{1}{2}$, 3, etc., merely multiply the ordinates of Eq. (1) by $\frac{1}{2}$, 3, etc., as the case may be. Likewise the curve $y = e^{-bx} \sin cx$ is readily drawn, since $\sin cx$ can be made from $\sin x$ by multiplying all abscissas of $\sin x$ by $1/c$.

CHAPTER IX

TRIGONOMETRIC EQUATIONS AND THE SOLUTIONS OF TRIANGLES

A. FURTHER TRIGONOMETRIC IDENTITIES

141. The Circle $\rho = a \cos \theta + b \sin \theta$.—In Sec. **73** an analytical proof was given of the fact that $\rho = a \cos \theta + b \sin \theta$ is the polar equation of a circle which passes through the pole, whose center is at the point $(\frac{1}{2}a, \frac{1}{2}b)$, and whose diameter is $\sqrt{a^2 + b^2}$. Any circle which passes through the pole may, however, be viewed as either a sine or cosine circle, of proper diameter, rotated into the desired position. Thus, it is always possible to write as alternative forms

$$\rho = a \cos \theta + b \sin \theta, \tag{1}$$

$$\rho = \sqrt{a^2 + b^2} \cos (\theta - \delta), \tag{2}$$

$$\rho = \sqrt{a^2 + b^2} \sin (\theta - \epsilon). \tag{3}$$

In any given problem it is possible to choose either positive or negative values for δ and ϵ, since the given circle may be viewed as the result of rotating a cosine or sine circle either forward or backward.

Illustration 1.—From the above we know that the equation $\rho = 6 \cos \theta + 8 \sin \theta$ represents a circle. The diameter of the circle is $\sqrt{a^2 + b^2} = \sqrt{6^2 + 8^2} = 10$, so that the equation of the circle may also be written $\rho = 10 \cos (\theta - \delta)$, where δ is the angle whose tangent is $\frac{b}{a} = \frac{8}{6} = 1.33$. From the table of tangents $\delta = 53°$, so that the equation of the circle may be written $\rho = 10 \cos (\theta - 53°)$.

Illustration 2.—Write the equation of the circle

$$\rho = -\cos \theta + \sqrt{3} \sin \theta \text{ in the form } \rho = c \cos (\theta - \delta)$$

and in the form

$$\rho = c \sin (\theta - \epsilon).$$

Here $a = -1$, $b = \sqrt{3}$, $c = \sqrt{a^2 + b^2} = 2$. The point C (Fig. 114) lies in the second quadrant. Hence $\delta = \tan^{-1} (\sqrt{3}/-1) = 120°$. Accordingly, ϵ is 30°. The desired equations are $\rho = 2 \cos (\theta - 120°)$ and $\rho = 2 \sin (\theta - 30°)$ (see Fig. 114).

Illustration 3.—Geometrical explanation:

In Fig. 115, $\rho_1 = a \cos\theta$ will be called the *a-circle* OA; $\rho_2 = b \sin\theta$ will be called the *b-circle* OB. For any value of the angle θ, draw radius vectors OM, ON, meeting the a- and b-circles respectively at the points M and N. If P be the point of intersection of MN produced with the circle whose diameter is the diagonal OC of the rectangle described on OA and OB, we shall show that $OM + ON = OP$, no matter in what direction OP be drawn.

Let the circle last mentioned be drawn, and project BC on OP. Since ONB and OPC are right angles, NP is the projection of BC $(= a)$ upon OP. But OM also is the projection of a $(= OA)$ upon OP.

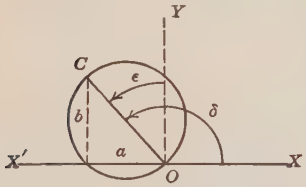


Fig. 114.—The circle $\rho = a \cos\theta + b \sin\theta$, $\rho = \sqrt{a^2 + b^2} \cos(\theta - \delta)$ or $\rho = \sqrt{a^2 + b^2} \sin(\theta - \epsilon)$. The figure is drawn for the case a negative, b positive.

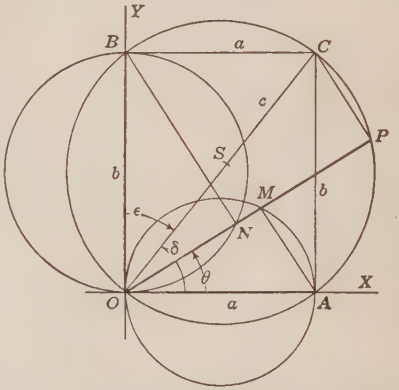


Fig. 115.—Combination of the circles $\rho = a \cos\theta$ and $\rho = b \sin\theta$ into a single circle $\rho = a \cos\theta + b \sin\theta$.

Hence $NP = OM$ because the projections of equal parallel lines on the same line are equal. Therefore, for all values of θ, $NP = \rho_1$ and $OP = ON + NP = \rho_2 + \rho_1$, which is the fact that was to be proved.

Illustration 4.—The radius vectors of two circles with diameters in the same direction add together to make a third circle, for $\rho = a \cos\theta + b \cos\theta = (a + b) \cos\theta$.

Exercises

1. Put the equation $\rho = 2 \cos\theta + 2\sqrt{3} \sin\theta$ in the form $\rho = c \cos(\theta - \delta)$ and find the value of δ.

2. Put the equation $\rho = 4 \cos\theta + \frac{4}{3}\sqrt{3} \sin\theta$ in the form $\rho = c \cos(\theta - \delta)$.

3. Put the equation $\rho = -4 \cos\theta - 4 \sin\theta$ in the form $\rho = c \sin(\theta - \epsilon)$.

4. Put the equation $\rho = 2\sqrt{3} \cos \theta + 2 \sin \theta$ in the form $\rho = c \cos (\theta - \delta)$.

5. Put the equation $\rho = 3 \cos \theta + 4 \sin \theta$ in the form $\rho = c \sin (\theta - \epsilon)$. Put the same equation in the form $\rho = c \cos (\theta - \delta)$.

6. Put the equation $\rho = 5 \cos \theta + 12 \sin \theta$ in the form $\rho = c \sin (\theta - \epsilon)$; also in the form $\rho = c \cos (\theta - \delta)$.

7. Put the equation $(x - 1)^2 + (y - 1)^2 = 2$ in the form $\rho = c \sin (\theta - \alpha)$ and determine c and α.

8. Put the equation $(x + 1)^2 + (y - \sqrt{3})^2 = 4$ in the form $\rho = c \sin (\theta - \alpha)$ and determine c and α.

9. Put the equation $(x + 1)^2 + (y + \sqrt{3})^2 = 4$ in the form $\rho = c \sin (\theta - \alpha)$ and determine c and α.

10. Find the maximum value of $\cos \theta - \sqrt{3} \sin \theta$, and determine the value of θ for which the expression is a maximum.

SUGGESTION: Call the expression ρ. The maximum value of ρ is the diameter of the circle $\rho = \cos \theta - \sqrt{3} \sin \theta$. The direction angle of the diameter is the value of α when the equation is put in the form $\rho = c \cos (\theta - \alpha)$.

11. Find the value of θ that renders $\rho = \frac{1}{3}\sqrt{3} \cos \theta - \frac{1}{2} \sin \theta$ a maximum and determine the maximum value of ρ.

12. Find the maximum value of $3 \cos t + 4 \sin t$.

142. Addition Formulas for the Sine and the Cosine.—From the preceding section, Eqs. (1), (2), and (3), we know that the equation of the circle OC (Fig. 116) may be written in any one of the forms

$$\rho = a \cos \theta + b \sin \theta, \qquad (1)$$

$$\rho = c \sin (\theta - \epsilon), \qquad (2)$$

$$\rho = c \cos (\theta - \delta). \qquad (3)$$

These equations hold identically in θ, whatever be the values of δ and ϵ. Indeed, the figure is drawn as it is, with the point C in the third quadrant, to emphasize the fact that there is no restriction of the values of δ and ϵ. Thus we have for all values of θ, δ, and ϵ.

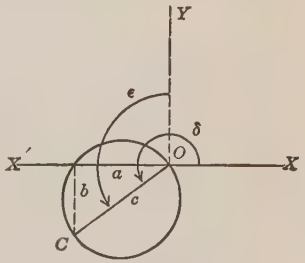


FIG. 116.—The circle $\rho = c \cos (\theta - \delta) = c \sin (\theta - \epsilon)$ used in the proof of the addition formulas.

$$\sin (\theta - \epsilon) = \frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta, \qquad (4)$$

$$\cos (\theta - \delta) = \frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta. \qquad (5)$$

In each of these equations $c = \sqrt{a^2 + b^2}$. The letters a and b stand for the rectangular coordinates of C, and hence represent either positive or negative numbers, depending upon the location of C.

Since Eqs. (4) and (5) are true for all values of θ, they are true when $\theta = 0°$ and when $\theta = 90°$.

First, let $\theta = 0°$ in Eq. (4). Then

$$\frac{a}{c} = \sin(-\epsilon) = -\sin\epsilon, \text{ by Sec. } \mathbf{62}, \tag{6}$$

Second, let $\theta = 90°$ in Eq. (4). Then

$$\frac{b}{c} = \sin(90° - \epsilon) = \cos\epsilon. \tag{7}$$

Substituting from (6) and (7) in Eq. (4) we have

$$\sin(\theta - \epsilon) = \sin\theta\cos\epsilon - \cos\theta\sin\epsilon. \tag{8}$$

In like manner, upon letting $\theta = 0°$ and $\theta = 90°$ in succession in Eq. (5), we have

$$\frac{a}{c} = \cos(-\delta) = \cos\delta, \text{ by Sec. } \mathbf{62}. \tag{9}$$

$$\frac{b}{c} = \cos(90 - \delta) = \sin\delta. \tag{10}$$

Substituting from (9) and (10) in Eq. (5) we obtain

$$\cos(\theta - \delta) = \cos\theta\cos\delta + \sin\theta\sin\delta. \tag{11}$$

Since these are true for all values of δ and ϵ, put $\delta = -\delta_1$ and $\epsilon = -\epsilon_1$. Then, by Sec. **62**, Eqs. (8) and (11) become

$$\sin(\theta + \epsilon_1) = \sin\theta\cos\epsilon_1 + \cos\theta\sin\epsilon_1. \tag{12}$$

$$\cos(\theta + \delta_1) = \cos\theta\cos\delta_1 - \sin\theta\sin\delta_1. \tag{13}$$

To aid in committing these four important formulas Eqs. (8), (11), (12) and (13) to memory, it is best to designate in each case the angles by α and β, and write Eqs. (12) and (13) in the form

$$\mathbf{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \tag{14}$$

$$\mathbf{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{15}$$

and also write Eqs. (8) and (11) in the form

$$\mathbf{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \tag{16}$$

$$\mathbf{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}. \tag{17}$$

The four formulas (Eqs. (14), (15), (16), and (17)) must be committed to memory. They are called the **addition formulas for the**

sine and cosine. The above demonstration shows that the addition formulas are true for *all values* of α and β.

By the above formulas it is possible to compute the sine and cosine of 75 and 15° from the following data:

$\sin 30° = \frac{1}{2}$. $\qquad \sin 45° = \frac{1}{2}\sqrt{2}$.

$\cos 30° = \frac{1}{2}\sqrt{3}$. $\qquad \cos 45° = \frac{1}{2}\sqrt{2}$.

Thus

$$\sin 75° = \sin (30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$$
$$= \tfrac{1}{2} \cdot \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3} \cdot \tfrac{1}{2}\sqrt{2} = \tfrac{1}{4}\sqrt{2}(\sqrt{3} + 1).$$

Likewise,

$$\sin 15° = \sin (45° - 30°) = \tfrac{1}{4}\sqrt{2}(\sqrt{3} - 1).$$

143. Addition Formula for the Tangent.—Dividing the members of Eq. (14), Sec. **142,** by the members of Eq. (15) we obtain

$$\tan (\alpha + \beta) = \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}. \quad (1)$$

Dividing numerator and denominator of the last fraction by $\cos \alpha \cos \beta$

$$\tan (\alpha + \beta) = \frac{\dfrac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \dfrac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\dfrac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \dfrac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \quad (2)$$

or

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \quad (3)$$

Likewise, it can be shown from Eqs. (16) and (17), Sec. **142,** that

$$\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}. \quad (4)$$

Equations (3) and (4) are the **addition formulas for the tangent.**

Exercises

1. Compute cos 75° and cos 15°.

2. Compute tan 75° and tan 15°.

3. Write in simple form the equation of the circle

$$\rho = \sin \theta + \cos \theta.$$

4. Put the equation of the circle $\rho = 3 \sin \theta + 4 \cos \theta$ in the form $\rho = c \sin (\theta + \theta_1)$ and find, from the tables or by the slide rule, the value of θ_1.

5. Derive a formula for $\cot(\alpha + \beta)$.

6. Prove $\cos(s + t)\cos(s - t) = \cos^2 s - \sin^2 t$.

7. Express in the form $c\cos(a - b)$ the binomial $3\cos a + 4\sin a$.

8. Express in the form $c\sin(a + b)$ the binomial $5\cos a + 12\sin a$.

9. Find the coordinates of the maximum point or crest of the sinusoid $y = \sin x + \sqrt{3}\cos x$. (First reduce the equation to the form $y = c\sin(x + \alpha)$.)

144. Functions of Composite Angles.—The sine, cosine, or tangent of the angles $(90° - \theta)$, $(90° + \theta)$, $(180° - \theta)$, $(180° + \theta)$, $(270° - \theta)$, $(270° + \theta)$ can be expressed in terms of functions of θ alone by means of the addition formulas of Secs. **142** and **143**. Thus, write

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{1}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{2}$$

Put $\alpha = 180°$, and $\beta = \pm\theta$; then Eqs. (1) and (2) become, respectively,

$$\sin(180° \pm \theta) = \mp\sin\theta. \tag{3}$$

$$\cos(180° \pm \theta) = -\cos\theta. \tag{4}$$

Also in Eqs. (1) and (2) put $\alpha = 90°$, and $\beta = \pm\theta$; then Eqs. (1) and (2) become, respectively,

$$\sin(90° \pm \theta) = \cos\theta. \tag{5}$$

$$\cos(90° \pm \theta) = \mp\sin\theta. \tag{6}$$

By division of Eq. (3) by Eq. (4), and of Eq. (5) by Eq. (6),

$$\tan(180° \pm \theta) = \pm\tan\theta, \tag{7}$$

$$\tan(90° \pm \theta) = \mp\cot\theta. \tag{8}$$

In a similar manner all of the results given in the following table may be proved to be true.

TABLE VII.—FUNCTIONS OF θ COUPLED WITH AN EVEN OR WITH AN ODD NUMBER OF RIGHT ANGLES

	$-\theta$	$90°-\theta$	$90°+\theta$	$180°-\theta$	$180°+\theta$	$270°-\theta$	$270°+\theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$

All of the above results can be included in two simple statements. For this purpose it is convenient to separate into different classes the composite angles that are made by coupling θ with an *odd* number of right angles, as $90° + \theta$, $\theta - 90°$, $270° - \theta$, $450° + \theta$, etc., and those composite angles that are made by coupling θ with an *even* number of right angles, as $180° + \theta$ $180° - \theta$, $360° - \theta$, $-\theta$, etc. Note that 0 is an even number, so that $-\theta$ or $0° - \theta$ falls into the second class of composite angles. We can then make the following statements:

Theorems on Functions of Composite Angles

I. *Any function of a composite angle made by coupling θ (by addition or subtraction) with an even number of right angles is numerically equal to the same function of the original angle θ.*

II. *Any function of a composite angle made by coupling θ (by addition or subtraction) with an odd number of right angles is numerically equal to the cofunction of the original angle θ.*

Note: The rule gives the *numerical* value. The sign to be attached must be determined, in each separate case, by a direct examination, as will be explained in the following illustrations. Although these rules are true regardless of the size of the angle θ, in the application of the rules, θ will usually be less than 90°, since the rules are chiefly used to express a function of an angle greater than 90° in terms of a function of an angle less than 90°. For example, the rule states that $\tan (180° + \theta)$ is numerically equal to $\tan \theta$. If θ be less than 90°, $(180° + \theta)$ lies in the third quadrant, and $\tan (180° + \theta)$ is positive. Hence $\tan (180° + \theta) = \tan \theta$. Similarly, the rule states that $\cos (270° - \theta)$ is numerically equal to $\sin \theta$. Moreover, if θ be less than 90°, $(270° - \theta)$ lies in the third quadrant, and $\cos (270° - \theta)$ is negative. Hence, $\cos (270° - \theta) = -\sin \theta$.

Exercises

1. Given $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\tan 30° = \frac{1}{3}\sqrt{3}$, $\cot 30° = \sqrt{3}$, find the sine, cosine, and tangent of each of the following angles by means of the above Theorems on Functions of Composite Angles: (*a*) 150°; (*b*) 210°; (*c*) 240°; (*d*) 300°; (*e*) 330°; (*f*) 120°; (*g*) 60°; (*h*) −30°.

2. Determine, by means of the above rules and the table in the back of the book, the values of sin 147°, cot 312°, cos 196°, tan 276°, tan 264°, sin 351°.

145. Angle That a Given Line Makes with Another Line.—The slope m of the straight line $y = mx + b$ is the tangent of the direction angle, that is, the tangent of the angle that the line makes with OX. *If L_1 and L_2 be any two lines in the plane,* the **angle that L_1 makes with L_2** *is the positive angle through which L_2 must be rotated about their point of intersection in order that L_2 may coincide with L_1.* Represent the direction angles of two straight lines

$$y = m_1x + b_1 \tag{1}$$
$$y = m_2x + b_2 \tag{2}$$

by the symbols θ_1 and θ_2. Then, through the intersection of the lines pass a line parallel to the OX-axis, as shown in Fig. 117.

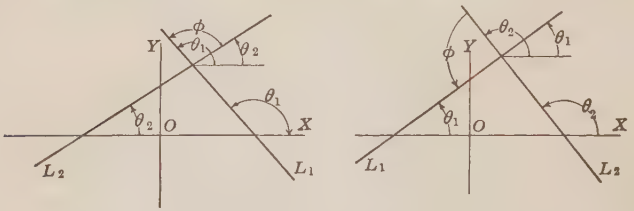


FIG. 117.—The angle ϕ that a line L_1 makes with L_2.

Call ϕ the angle that the line L_1 makes with L_2; that is, the positive angle through which L_2, considered as the initial line, must be turned to coincide with the terminal position given by L_1. If $\theta_1 > \theta_2$, then $\phi = \theta_1 - \theta_2$, but if $\theta_2 > \theta_1$, then $\phi = 180° - (\theta_2 - \theta_1)$. In either case (by Eq. (7), Sec. **144**, and (3), Sec. **62**)

$$\tan \phi = \tan (\theta_1 - \theta_2), \tag{3}$$

that is,

$$\tan \phi = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}, \tag{4}$$

or

$$\tan \phi = \frac{m_1 - m_2}{1 + m_1m_2}. \tag{5}$$

The condition under which the given lines (Eqs. (1) and (2)) are parallel is, obviously, that

$$\mathbf{m}_1 = \mathbf{m}_2. \tag{6}$$

Thus, the lines $y = 5x + 7$ and $y = 5x - 11$ are parallel.

The condition under which the given lines (Eqs. (1) and (2)) are perpendicular to each other is that $\tan \phi$ shall become infinite; that is, that the denominator of Eq. (5) shall vanish. Hence, the condition of perpendicularity is

$$1 + m_1 m_2 = 0,$$

or

$$\mathbf{m}_1 = -\frac{1}{\mathbf{m}_2}. \tag{7}$$

Therefore, *in order that two lines may be perpendicular to each other, the slope of one line must be the negative reciprocal of the slope of the other line.*

Thus the lines $y = \frac{2}{3}x - 4$ and $y = -\frac{3}{2}x + 2$ are perpendicular.

Exercises

1. Find the tangent of the angle that the first line makes with the second line of each set:

(*a*) $y = 2x + 3$,	$y = x + 2$.
(*b*) $y = 3x - 3$,	$y = 2x + 1$.
(*c*) $y = 4x + 5$,	$y = 3x - 1$.
(*d*) $y = 10x + 1$,	$y = 11x - 1$.

2. Find the angle that the first line of each pair makes with the second:

(*a*) $y = x + 5$,	$y = -x + 5$.
(*b*) $y = \frac{1}{2}x + 6$,	$y = -2x$.
(*c*) $y = 2x + 4$,	$y = x + 1$.
(*d*) $2x + 3y = 1$,	$\frac{2}{3}x = y - 1$.
(*e*) $2x + 4y = 3$,	$3x + 6y = 7$.
(*f*) $2x + 4y = 3$,	$6x - 3y = 7$.

3. Find the angle, in each of the following cases, that the first line makes with the second:

(*a*) $y = \frac{x}{\sqrt{3}} + 4$,	$y = \sqrt{3}\,x + 2$.
(*b*) $y = \frac{x}{\sqrt{3}} + 1$,	$y = \sqrt{3}\,x - 4$.
(*c*) $y = \sqrt{3}\,x - 6$,	$y = \sqrt{3}\,x - 3$.

4. Find the angle that $2y - 6x + 7 = 0$ makes with $y + 2x + 7 = 0$ and also the angle that the second line makes with the first.

146. The Functions of the Double Angle.—The addition formulas for the sine, cosine, and tangent reduce to formulas of great importance for the special case $\beta = \alpha$.

Thus

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha,$$

or

$$\sin 2\alpha = 2\sin\alpha\cos\alpha. \tag{1}$$

Also

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha,$$

which may be written in the three forms:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha, \tag{2}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1, \tag{3}$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha. \tag{4}$$

Equations (3) and (4) are obtained from Eq. (2) by substituting, respectively, $\sin^2\alpha = 1 - \cos^2\alpha$ and $\cos^2\alpha = 1 - \sin^2\alpha$.

Equations (3) and (4) are frequently useful in the forms:

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}, \tag{5}$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}. \tag{6}$$

Again,

$$\tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha},$$

or

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}. \tag{7}$$

147. The Functions of the Half Angle.—From Eqs. (6) and (5) of Sec. **146** we obtain, after replacing α by $u/2$ and extracting the square root,

$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{(1 - \cos u)}{2}}, \tag{1}$$

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{(1 + \cos u)}{2}}. \tag{2}$$

Dividing Eq. (1) by (2), we obtain

$$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{1+\cos u}} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}. \quad (3)$$

Equations (1), (2), and (3) have many important applications in mathematics. As a simple example, note that the functions of 15° may be computed when the functions of 30° are known. Thus,

$$\cos 30° = \tfrac{1}{2}\sqrt{3}.$$

Therefore,

$$\sin 15° = \sqrt{\frac{(1-\cos 30°)}{2}} = \sqrt{\tfrac{1}{2} - \tfrac{1}{4}\sqrt{3}}.$$

Also,

$$\cos 15° = \sqrt{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{3}}.$$

Likewise, by Eq. (3),

$$\tan 15° = \frac{1 - \frac{1}{2}\sqrt{3}}{\frac{1}{2}} = 2 - \sqrt{3}.$$

Exercises

1. Compute sin 60° from the sine and cosine of 30°.
2. Compute sine, cosine, and tangent of 22½°.
3. If $\sin x = \frac{2}{5}$, find the numerical value of sin $2x$, cos $2x$, and tan $2x$.
4. Show by expanding sin $(x + 2x)$ that $\sin 3x = 3\sin x - 4\sin^3 x$.
5. Prove $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.
6. Show that $\sin 2\theta/\sin\theta - \cos 2\theta/\cos\theta = \sec\theta$.
7. Show that
$$\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2 = 1 + \sin\theta.$$
8. Show that $\cos 2\theta(1 + \tan 2\theta \tan\theta) = 1$.
9. If $\sin A = \frac{3}{5}$, calculate $\sin(A/2)$.
10. Prove that $\tan(\pi/4 + \theta) = (1 + \tan\theta)/(1 - \tan\theta)$.
11. Prove that $\tan(\pi/4 - \theta) = (1 - \tan\theta)/(1 + \tan\theta)$.
12. Show that $\sec\theta + \tan\theta = \dfrac{1 + \sin\theta}{\cos\theta}$.
13. Show that $\dfrac{1 + 2\sin a\cos a}{\cos^2 a - \sin^2 a} = \dfrac{\cos a + \sin a}{\cos a - \sin a}$.
14. Show that $\sec\theta + \tan\theta = \tan\left[\dfrac{\pi}{4} + \dfrac{\theta}{2}\right]$.

15. Show that $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.

16. Prove that $\cos (s + t) \cos (s - t) + \sin (s + t) \sin (s - t) = \cos 2t$.

17. Investigate cases in which $u/2$ falls in all four quadrants, and justify the omission of plus and minus signs in the last two expressions of Eq. (3) above.

148. Sums and Differences of Sines and of Cosines Expressed as Products.—The following formulas, which permit the substitution of a product for a sum of two sines or of two cosines, are important in many transformations in mathematics, especially in the calculus. They are immediately derivable from the addition formulas. Thus, by the addition formulas (Eqs. (14) and (16), Sec. **142**), we obtain

$$\sin (a + b) + \sin (a - b) = 2 \sin a \cos b.$$

Likewise, by subtraction of the same formulas,

$$\sin (a + b) - \sin (a - b) = 2 \cos a \sin b.$$

By the addition and subtraction, respectively, of the addition formulas for the cosine there results

$$\cos (a + b) + \cos (a - b) = 2 \cos a \cos b.$$
$$\cos (a + b) - \cos (a - b) = -2 \sin a \sin b.$$

These formulas can be written

$$\sin a \cos b = \tfrac{1}{2} [\sin (a + b) + \sin (a - b)]. \quad (1)$$
$$\cos a \sin b = \tfrac{1}{2} [\sin (a + b) - \sin (a - b)]. \quad (2)$$
$$\cos a \cos b = \tfrac{1}{2} [\cos (a + b) + \cos (a - b)]. \quad (3)$$
$$\sin a \sin b = -\tfrac{1}{2} [\cos (a + b) - \cos (a - b)]. \quad (4)$$

Represent $(a + b)$ by α and $(a - b)$ by β. Then $a = (\alpha + \beta)/2$ and $b = (\alpha - \beta)/2$.

Hence the above formulas become

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}. \quad (5)$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}. \quad (6)$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}. \quad (7)$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}. \quad (8)$$

The principal use of these formulas is in certain transformations in the calculus. A minor use is in adapting certain formulas to logarithmic work by replacing sums and differences by products.

These formulas should not be committed to memory. They can be derived in a moment when needed by recalling their connection with the addition formulas. Equation (2) is really contained in Eq. (1), for by Eq. (1)

$$\begin{aligned}\cos a \sin b &= \sin b \cos a \\ &= \tfrac{1}{2}[\sin (b + a) + \sin (b - a)] \\ &= \tfrac{1}{2}[\sin (a + b) - \sin (a - b)],\end{aligned}$$

since

$$\sin (-\theta) = -\sin \theta.$$

Exercises

Express as the sum or difference of sines or cosines:

1. $\sin 5x \cos 2x$.
2. $\cos 3x \sin 7x$.
3. $\cos 4x \cos x$.
4. $\sin 5x \sin 2x$.
5. $\sin 3x \cos 5x$.
6. $\sin 3x \sin 7x$.
7. $\cos 3x \cos 8x$.
8. $\cos 5x \sin 2x$.
9. $\sin 3x \cos 10x$.
10. $\cos 2x \cos 6x$.

B. PLANE TRIANGLES: CONDITIONAL EQUATIONS

149. Law of Sines.—The first of the conditional equations pertaining to the oblique triangle is a proportion connecting the

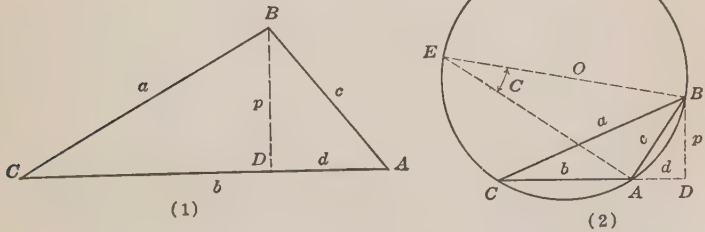


Fig. 118.—Derivation of the law of sines and the law of cosines.

sines of the three angles of the triangle with the lengths of the respective sides lying opposite. Call the angles of the triangle A, B, C, and indicate the opposite sides by the small letters

a, b, c, respectively. From the vertex of any angle, drop a perpendicular p upon the opposite side, meeting the latter (produced if necessary) at D. Then, by the properties of right triangles, we have, in either Fig. 118, (1) or Fig. 118 (2),

$$p = c \sin DAB. \tag{1}$$

From ΔBDC,

$$p = a \sin C. \tag{2}$$

But,

$$\begin{aligned} \sin DAB &= \sin A \qquad \text{(Fig. 118, (1))} \\ &= \sin (180° - A) \quad \text{(Fig. 118, (2))} \\ &= \sin A. \end{aligned}$$

Therefore,

$$p = c \sin A = a \sin C, \tag{3}$$

or

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \tag{4}$$

In like manner, by dropping a perpendicular from A upon a, we can prove

$$\frac{b}{\sin B} = \frac{c}{\sin C}. \tag{5}$$

Therefore,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{6}$$

Stated in words, the formula says: *In any oblique triangle the sides are proportional to the sines of the opposite angles.*

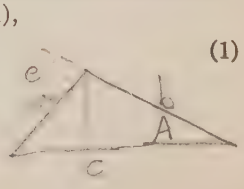
Geometrically.—Calling each of the ratios in Eq. (6) $2R$, it is seen from Fig. 118, (2) that R is the radius of the circumscribed circle since $c/\sin C = 2R$ can be deduced from the triangle BAE. Similar construction can be made for the angle B or A.

150. Law of Cosines.—From plane geometry we have the theorem: *The square of any side opposite an acute angle of an oblique triangle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other side on it.* Thus, in Fig. 118, (1),

$$a^2 = b^2 + c^2 - 2bd. \tag{1}$$

But

$$d = c \cos A.$$



Therefore,

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{2}$$

Likewise, we learn from geometry that the *square of any side opposite an obtuse angle of an oblique triangle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides by the projection of the other on it.* Thus, in Fig. 118, (2),

$$a^2 = b^2 + c^2 + 2bd. \tag{3}$$

But

$$d = c \cos DAB = c \cos (180 - A) = -c \cos A.$$

Therefore Eq. (3) becomes

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A.} \tag{4}$$

This is the same as Eq. (2), so that the trigonometric form of the geometrical theorem is the same whether the side first named is opposite an acute or opposite an obtuse angle.

In the same way we may show that in any triangle

$$\mathbf{b^2 = c^2 + a^2 - 2ca \cos B.} \tag{5}$$
$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C.} \tag{6}$$

Second Proof.—Independently of the theorem from plane geometry, we note from Fig. 118, (1)

$$\begin{aligned} a^2 &= (b - d)^2 + p^2 = (b - d)^2 + c^2 - d^2 \\ &= b^2 + c^2 - 2bd \\ &= b^2 + c^2 - 2bc \cos A. \end{aligned}$$

From 118, (2)

$$\begin{aligned} a^2 &= (b + d)^2 + p^2 = (b + d)^2 + c^2 - d^2 \\ &= b^2 + c^2 + 2bd \\ &= b^2 + c^2 + 2bc \cos DAB \\ &= b^2 + c^2 - 2bc \cos A, \end{aligned}$$

since $DAB = 180° - A$ and $\cos (180° - A) = -\cos A$.

Third Proof.—Since any side of an oblique triangle is the sum of the projections of the other two sides upon it, the angles of projection being the angles of the triangle, we have

$$\begin{aligned} \mathbf{a &= b \cos C + c \cos B.} \\ \mathbf{b &= c \cos A + a \cos C.} \\ \mathbf{c &= a \cos B + b \cos A.} \end{aligned} \tag{7}$$

Multiply the first of these equations by a, the second by b, the third by c, and subtract the second and third from the first. The result is

$$\begin{aligned} a^2 - b^2 - c^2 &= ab\cos C + ca\cos B \\ &\quad - bc\cos A - ab\cos C \\ &\quad - ca\cos B - bc\cos A \\ &= -2bc\cos A, \end{aligned}$$

or

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

151. Law of Tangents.—An important relation results if we take Eq. (5), Sec. **149** by composition and division. First write the law of sines in the form

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \tag{1}$$

Then, by composition and division, the sum of the first antecedent and consequent is to their difference as the sum of the second antecedent and consequent is to their difference; that is

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}. \tag{2}$$

Expressing the sums and the difference on the right-hand side of Eq. (2) as products by means of Eqs. (5) and (6) of Sec. **148,** we obtain

$$\frac{a+b}{a-b} = \frac{2\sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}(A-B)}{2\cos \tfrac{1}{2}(A+B)\sin \tfrac{1}{2}(A-B)}, \tag{3}$$

or simplifying and replacing the ratio of sine to cosine by the tangent, we obtain

$$\frac{\mathbf{a+b}}{\mathbf{a-b}} = \frac{\tan \tfrac{1}{2}(\mathbf{A+B})}{\tan \tfrac{1}{2}(\mathbf{A-B})}. \tag{4}$$

In like manner it follows that

$$\frac{\mathbf{b+c}}{\mathbf{b-c}} = \frac{\tan \tfrac{1}{2}(\mathbf{B+C})}{\tan \tfrac{1}{2}(\mathbf{B-C})}, \tag{5}$$

$$\frac{\mathbf{c+a}}{\mathbf{c-a}} = \frac{\tan \tfrac{1}{2}(\mathbf{C+A})}{\tan \tfrac{1}{2}(\mathbf{C-A})}. \tag{6}$$

Expressed in words: *In any triangle, the sum of two sides is to their difference as the tangent of half the sum of the angles opposite is to the tangent of half of their difference.*

Geometrical Proof.—From any vertex of the triangle as center, say C, draw a circle of radius equal to the shortest of the two sides of the triangle meeting at C, as in Fig. 119. Let the circle meet the side a at R and the same side produced at E. Draw AE and AR. Call the angles at A, α, and β, as shown. Then $BE = a + b$ and $BR = a - b$. Also

$$\alpha + \beta = A,$$

and $\angle CRA = \beta + B$ (the external angle of a triangle RAB is equal to the sum of the two opposite interior angles), or

$$\alpha - \beta = B.$$

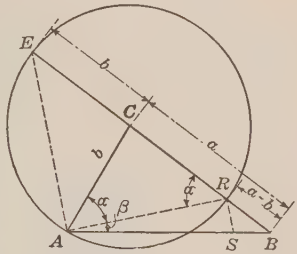


FIG. 119.—Geometrical derivation of law of tangents.

Therefore

$$\alpha = \tfrac{1}{2}(A + B).$$
$$\beta = \tfrac{1}{2}(A - B).$$

Draw RS $\parallel$ to EA. $\angle EAR = \angle ARS = 90°$. By similar triangles

$$\frac{BE}{BR} = \frac{AE}{SR}$$
$$= \frac{AE}{AR} \div \frac{SR}{AR}.$$

But $BE = a + b$ and $BR = a - b$, while

$$\frac{AE}{AR} = \tan \alpha \text{ and } \frac{SR}{AR} = \tan \beta.$$

Therefore,

$$\frac{a + b}{a - b} = \frac{\tan \tfrac{1}{2}(A + B)}{\tan \tfrac{1}{2}(A - B)}.$$

152. The following special formulas are readily deduced from the sine formulas and are sometimes useful as check formulas in computation. They are closely related to the law of tangents. From the proportion

$$a : b : c = \sin A : \sin B : \sin C$$

by composition

$$\frac{c}{a + b} = \frac{\sin C}{\sin A + \sin B}.$$

Now by Sec. **146,** Eq. (1) and Sec. **148,** Eq. (5) this may be written

$$\frac{c}{a+b} = \frac{2 \sin \frac{1}{2}C \cos \frac{1}{2}C}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}.$$

Since $C = 180° - (A + B)$, therefore

$$\frac{C}{2} = 90° - \tfrac{1}{2}(A+B), \text{ and } \cos \frac{C}{2} = \sin \tfrac{1}{2}(A+B).$$

Hence,

$$\frac{\mathbf{c}}{\mathbf{a+b}} = \frac{\sin \frac{1}{2}\mathbf{C}}{\cos \frac{1}{2}(\mathbf{A}-\mathbf{B})} = \frac{\cos \frac{1}{2}(\mathbf{A}+\mathbf{B})}{\cos \frac{1}{2}(\mathbf{A}-\mathbf{B})}. \tag{1}$$

In like manner it can be proved that

$$\frac{\mathbf{c}}{\mathbf{a-b}} = \frac{\sin \frac{1}{2}(\mathbf{A}+\mathbf{B})}{\sin \frac{1}{2}(\mathbf{A}-\mathbf{B})}. \tag{2}$$

Both Eqs. (1) and (2) can be readily deduced geometrically from Fig. 119.

153. The *s*-formulas.—The cosine formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

can be written in the forms

$$a^2 = (b+c)^2 - 2bc(1 + \cos A), \tag{1}$$
$$a^2 = (b-c)^2 + 2bc(1 - \cos A), \tag{2}$$

by adding $(+2bc)$ and $(-2bc)$ to the right-hand member in each case. But now we know from Sec. **147,** Eqs. (1) and (2), that

$$1 + \cos A = 2 \cos^2 \left(\frac{A}{2}\right).$$
$$1 - \cos A = 2 \sin^2 \left(\frac{A}{2}\right).$$

Therefore Eqs. (1) and (2) above become

$$a^2 = (b+c)^2 - 4bc \cos^2 \left(\frac{A}{2}\right). \tag{3}$$

$$a^2 = (b-c)^2 + 4bc \sin^2 \left(\frac{A}{2}\right). \tag{4}$$

Writing these in the form

$$4bc \sin^2 \left(\frac{A}{2}\right) = a^2 - (b-c)^2, \tag{5}$$

$$4bc \cos^2 \left(\frac{A}{2}\right) = (b+c)^2 - a^2, \tag{6}$$

and dividing the members of Eq. (5) by the members of Eq. (6), we obtain

$$\tan^2\left(\frac{A}{2}\right) = \frac{a^2 - (b-c)^2}{(b+c)^2 - a^2}. \tag{7}$$

Factoring the numerator and denominator we obtain

$$\tan^2\left(\frac{A}{2}\right) = \frac{(a+b-c)(a-b+c)}{(b+c+a)(b+c-a)}. \tag{8}$$

Let the perimeter of the triangle be represented by $2s$, that is, let

$$a + b + c = 2s.$$

Hence, subtracting $2c$, $2b$, and $2a$ in turn,

$$a + b - c = 2s - 2c \text{ (subtracting } 2c).$$
$$a - b + c = 2s - 2b \text{ (subtracting } 2b).$$
$$b + c - a = 2s - 2a \text{ (subtracting } 2a).$$

Therefore, Eq. (8) becomes

$$\tan^2\left(\frac{A}{2}\right) = \frac{(s-b)(s-c)}{s(s-a)}. \tag{9}$$

Let

$$\frac{(s-a)(s-b)(s-c)}{s} = r^2. \tag{10}$$

Then

$$\tan^2\left(\frac{A}{2}\right) = \frac{r^2}{(s-a)^2},$$

or

$$\tan\left(\frac{A}{2}\right) = \frac{r}{(s-a)}. \tag{11}$$

Likewise,

$$\tan\left(\frac{B}{2}\right) = \frac{r}{(s-b)}. \tag{12}$$

$$\tan\left(\frac{C}{2}\right) = \frac{r}{(s-c)}. \tag{13}$$

Geometrically.—These formulas may be found by means of the diagram (Fig. 120). Let the circle O be inscribed in the triangle ABC; its center is located at the intersection of the bisectors of the internal angles of the triangle. Let its radius be r. $AT_1 = AT_3$, $BT_2 = BT_3$, $CT_1 = CT_2$, and, since $2s = a + b + c$, it follows that one way of writing the value of s is

$$s = BT_2 + T_2C + AT_1.$$

Therefore

$$AT_1 = s - a.$$

Hence it follows that

$$\tan\left(\frac{A}{2}\right) = \frac{r}{(s-a)}. \tag{14}$$

Since this result is the same as Eq. (11) above, it proves that the r of Eq. (10) is the radius of the inscribed circle, and therefore proves

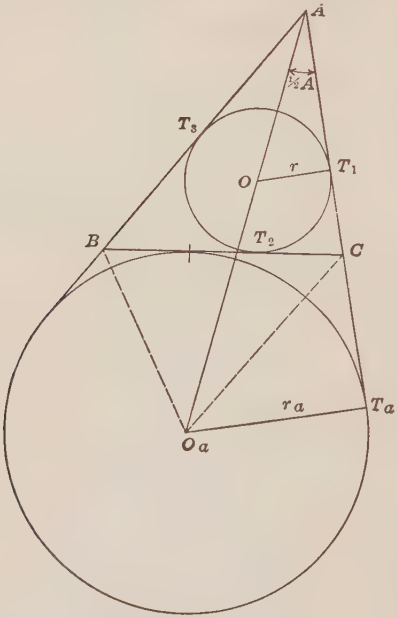


FIG. 120.—Geometrical derivation of the s-formulas.

that the radius of the inscribed circle may be expressed by the formula

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{15}$$

a fact that is usually proved in textbooks on plane geometry.

154. Miscellaneous Formulas for Oblique Triangles.—The following formulas are given without proof. They are occasionally useful for reference, although no use will be made of them in this book. The following notation is used: The three sides of

the oblique triangle are named a, b, c, and the angles opposite these A, B, C, respectively. The semi-perimeter of the triangle is s, or $2s = a + b + c$. The radius of the circumscribed circle is R, that of the inscribed circle is r, and the radii of the escribed circles are r_a, r_b, r_c, tangent, respectively, to the sides a, b, c of the given triangle. K stands for the area of the triangle.

$$s = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \tag{1}$$

$$s - c = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C \tag{2}$$

and analogs for $s - a$ and $s - b$.

$$r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \tag{3}$$

$$r_c = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C \tag{4}$$

and analogs for r_a and r_b.

$$r_a = s \tan \tfrac{1}{2}A,\ r_b = s \tan \tfrac{1}{2}B,\ r_c = s \tan \tfrac{1}{2}C. \tag{5}$$

$$2K = ab \sin C = bc \sin A = ca \sin B. \tag{6}$$

$$K = 2R^2 \sin A \sin B \sin C = \frac{abc}{4R}. \tag{7}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \tag{8}$$

$$K = rs = r_a(s-a) = r_b(s-b) = r_c(s-c). \tag{9}$$

$$K^2 = rr_ar_br_c. \tag{10}$$

C. NUMERICAL SOLUTION OF OBLIQUE TRIANGLES

155. An oblique triangle possesses six elements, namely, the three sides and the three angles. If any three of these six magnitudes be given (except the three angles), the triangle is determinate, or may be constructed by the methods explained in plane geometry; it will also be found that if any three of these six magnitudes be given, the other three may be computed by the formulas of trigonometry, provided the given parts include at least one side.

It is convenient to divide the solution of triangles into four cases, as follows:

I. Given two angles and one side.
II. Given two sides and an angle opposite one of them.
III. Given two sides and the included angle.
IV. Given the three sides.

The solution of these cases with appropriate checks will now be given. The best arrangement of the work of computation usually consists in writing the data and the computed results in the left margin of a sheet of ruled letter paper (8½ by 11 in.) and placing the computation in the body of the sheet. Every entry should be carefully labeled and computed results should be enclosed in square brackets. All work should be done on ruled paper and *invariably in ink*. Special calculation sheets (forms *M*2 and *M*7) have been prepared for the use of students. Neatness and systematic arrangement of the work and proper checking are more important than rapidity of calculation

156. Computer's Rules.—The following computer's rules are useful to remember in logarithmic work:

Last Digit Even.—When it becomes necessary to discard a 5 that terminates any decimal, increase by unity the last digit retained if it be an odd digit, but leave it unchanged if it be an even digit; that is, *keep the last digit retained even*. Thus, $\log \pi = 0.4971$; hence, write $\frac{1}{2} \log \pi = 0.2486$. Also, $\log \sin 18° 5' = 9.4900 +$ (correction) $19.5 = 9.4920$.

Of course, if the discarded figure be greater than 5, the last digit retained should be increased by 1, while if the discarded figure be less than 5, the last digit retained should be left unchanged.

Functions of Angles in Second Quadrant.—In finding from the table any function of an angle greater than 100° (but $< 180°$) *replace the first two digits of the number of degrees in the angle by their sum and take the cofunction of the result.* The method is valid because it is equivalent to the subtraction of 90° from the angle. By Sec. **144** this always gives the correct numerical value of the function. The algebraic sign should be taken into account separately. Thus, $\sin 157° 32' 7'' = \cos 67° 32' 7''$. In case of an angle between 90 and 100°, ignore the first digit and proceed in the same way. Thus,

$$\tan 97° 57' 42'' = -\cot 7° 57' 42''.$$

157. Case I.—*Given two angles and one side*, as A, B, and c.

1. To find C, use the relation $A + B + C = 180°$.
2. To find a and b, use the law of sines (Sec. **149**).
3. To check results, apply the check, Eqs. (1) or (2), Sec. **152.**

Illustration: In the oblique triangle, let $c = 1492$, $A = 49° 52'$, $B = 27° 15'$. It is required to compute C, a, b.

The following form of work is self-explanatory. It should be noted that the process of work and the meaning of each number entering the calculation is properly indicated or labeled in the work.

Numerical Work

Given:

$c = 1492.$

$A = 49° 52'.$

$B = 27° 15'.$

To find a, b, and C.

Formulas:

$$C = 180 - (A + B) = (102° 53').$$

$$a = \frac{c \sin A}{\sin C}.$$

$$b = \frac{c \sin B}{\sin C}.$$

Work:

$\log \sin A = 9.8834 - 10$

$\log c = 3.1738$

$\log \sin B = 9.6608 - 10$

$\log \sin C = 9.9889 - 10$

$\log a = 3.0683$

$\log b = 2.8457$

$a = [1170].$

$b = [701].$

Check:

$c - b = 791$

$C + B = 130° 8'$

$C - B = 75° 38'$

$\frac{1}{2}(C + B) = 65° 4'$

$\frac{1}{2}(C - B) = 37° 49'$

Check formula:

$$\frac{a}{c - b} = \frac{\sin \frac{1}{2}(C + B)}{\sin \frac{1}{2}(C - B)}.$$

$\log a = 3.0683$

$\log c - b = 2.8982$

$\log \frac{a}{c - b} = 0.1701$

$\log \sin \frac{1}{2}(C + B) = 9.9575 - 10$

$\log \sin \frac{1}{2}(C - B) = 9.7875 - 10$

$\log \frac{\sin \frac{1}{2}(C + B)}{\sin \frac{1}{2}(C - B)} = 0.1700$

} Check.

Examples

Find the remaining parts, given:

1. $A = 47° 20'$, $B = 32° 10'$, $a = 739.$
2. $B = 37° 38'$, $C = 77° 23'$, $b = 1224.$
3. $B = 25° 2'$, $C = 105° 17'$, $b = 0.3272.$
4. $C = 19° 35'$, $A = 79° 47'$, $c = 56.47.$

158. Case II.—*Given two sides and an angle opposite one of them, as* a, b, and A.

1. To find B, use the law of sines (Sec. **149**).
2. To find C, use the equation $A + B + C = 180°$.
3. To find c use the law of sines.
4. To check, apply the check Eqs. (1) or (2), Sec. **152**.

When an angle as B, above, is determined from its sine, it admits of two values, which are supplementary to each other. There may be, therefore, two solutions to a triangle in Case II. The solutions are illustrated in Fig. 121.

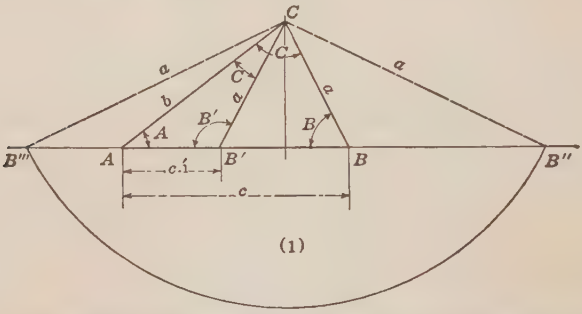


FIG. 121.—Case II of triangles, for one, two, and impossible solutions.

In case one of the two values of B when added to the given angle A gives a sum greater than two right angles, this value of B must be discarded, and but one solution exists. If a be less than the perpendicular distance from C to c, no solution is possible.

Illustration: Solve the triangle if $a = 345$, $b = 534$, and $A = 25° 25'$.

The solution is readily understood from the following work.

Numerical Work

Given:

$a = 345.$

$b = 534.$

$A = 25° 25'.$

To find c, B, and C.

Formulas:

$$\sin B = \frac{b \sin A}{a}.$$

$$C = 180 - (A + B).$$

$$c = \frac{a \sin C}{\sin A}.$$

Work:

$\log b$	$= 2.7275$	
$\log \sin A$	$= 9.6326 - 10$	
$\log a$	$= 2.5378$	
$\log \sin B$	$= 9.8223 - 10$	
B	$= [41° 37'.1]$ or	$[138° 22'.9]$
$A + B$	$= 67° 2'.1$	$163° 47'.9$
C	$= [112° 57'.9]$ or	$[16° 12'.1]$
$\log a$	$= 2.5378$	2.5378
$\log \sin C$	$= 9.9641 - 10$	$9.4456 - 10$
$\log \sin A$	$= 9.6326 - 10$	$9.6326 - 10$
$\log c$	$= 2.8693$	2.3508
c	$= [740.1]$	$[224.3]$

$$\frac{b}{c-a} = \frac{\sin \frac{1}{2}(C+A)}{\sin \frac{1}{2}(C-A)} = Q$$

$c - a$	$= 395.1$
$C + A$	$= 138° 22'.9$
$C - A$	$= 87° 32'.9$
$\frac{1}{2}(C + A)$	$= 69° 11'.5$
$\frac{1}{2}(C - A)$	$= 43° 46'.5$
$\log b$	$= 2.7275$
$\log (c - a)$	$= 2.5967$
$\log Q$	$= 0.1308$
$\log \sin \frac{1}{2}(C + A)$	$= 9.9707 - 10$
$\log \sin \frac{1}{2}(C - A)$	$= 9.8400 - 10$
$\log Q$	$= 0.1307$

Check:

$$\frac{a}{b-c} = \frac{\sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B-C)} = Q'$$

$b - c$	$= 309.7$
$B + C$	$= 154° 35'$
$B - C$	$= 122° 10'.8$
$\frac{1}{2}(B + C)$	$= 77° 17'.5$
$\frac{1}{2}(B - C)$	$= 61° 5'.4$
$\log a$	$= 2.5378$
$\log (b - c)$	$= 2.4910$
$\log Q'$	$= 0.0468$
$\log \sin \frac{1}{2}(B + C)$	$= 9.9892 - 10$
$\log \sin \frac{1}{2}(B - C)$	$= 9.9422 - 10$
$\log Q'$	$= 0.0470$

Examples

Compute the unknown parts in each of the following triangles:

1.	$a = 0.8,$	$b = 0.6,$	$B = 40° 15'.$
2.	$a = 8.81,$	$b = 11.87,$	$A = 19° 9'.$
3.	$b = 81.05,$	$c = 98.75,$	$C = 99° 19'.$
4.	$c = 50.37,$	$a = 58.11,$	$C = 78° 13'.$
5.	$a = 1213,$	$b = 1156,$	$B = 94° 15'.$

159. Case III.—*Given two sides and the included angle,* as a, b, C.

1. To find $A + B$, use $A + B = 180° - C$.

2. To find A and B, compute $(A - B)/2$ by the law of tangents (Sec. **151,** Eq. (4)), then $A = (A + B)/2 + (A - B)/2$ and $B = (A + B)/2 - (A - B)/2$.

3. To find c, use law of sines (Sec. **149**).
4. To check, use law of sines.

Illustration: Given $a = 1033$, $b = 635$, $C = 38° 36'$.

Numerical Work

Given:
$a = 1033.$
$b = 635.$
$C = 38° 36'.$

To find c, A, and B.
Formulas:

$$A + B = 180 - C = 141° 24'.$$

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B).$$

$$c = \frac{a \sin C}{\sin A}.$$

Work:

$a - b$	$= 398$
$a + b$	$= 1668$
$\frac{1}{2}(A + B)$	$= 70° 42'$
$\log (a - b)$	$= 2.5999$
$\log \tan \frac{1}{2}(A + B)$	$= 0.4557$
$\log (a + b)$	$= 3.2222$
$\log \tan \frac{1}{2}(A - B)$	$= 9.8334 - 10$
$\frac{1}{2}(A - B)$	$= 34° 16.3'$
A	$= [104° 58.3']$
B	$= [36° 25.7']$
$\log a$	$= 3.0141$
$\log \sin C$	$= 9.7951 - 10$
$\log \sin A$	$= 9.9850 - 10$
$\log c$	$= 2.8242$
	$c = [667.1]$

Check:

$$c = \frac{b \sin C}{\sin B}$$

$\log b = 2.8028$
$\log \sin C = 9.7951 - 10$
$\log \sin B = 9.7737 - 10$
$\log c = 2.8242$
$c = [667.1]$

Examples

Compute the unknown parts in each of the following triangles:

1. $a = 78.9$, $b = 68.7$, $C = 78° 10'$.
2. $c = 70.16$, $a = 39.14$, $B = 16° 16'$.
3. $b = 1781$, $c = 982.7$, $A = 123° 16'$.

160. Case IV.—*Given the three sides.*

1. To find the angles, use the s-formulas (Sec. **153,** Eqs. (11), (12), and (13)).

2. To check, use $A + B + C = 180°$.

Illustration: Given $a = 455$, $b = 566$, $c = 677$, find A, B, and C.

Numerical Work

Given:

$a = 455.$
$b = 566.$
$c = 677.$

To find A, B, and C.

Formulas:

$$\tan \tfrac{1}{2}A = \frac{r}{s-a},$$

$$\tan \tfrac{1}{2}B = \frac{r}{s-b},$$

$$\tan \tfrac{1}{2}C = \frac{r}{s-c},$$

$$\text{where } r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Work:

$2s = 1698$
$s = 849$
$s - a = 394$
$s - b = 283$
$s - c = 172$
$2s^1 = 1698$
$\log (s-a) = 2.5955$
$\log (s-b) = 2.4518$
$\log (s-c) = 2.2355$
$\log s = 2.9289$
$\log r^2 = 4.3539$
$\log r = 2.1770$
$\log \tan \tfrac{1}{2}A = 9.5815 - 10$
$\log \tan \tfrac{1}{2}B = 9.7252 - 10$
$\log \tan \tfrac{1}{2}C = 9.9415 - 10$
$\tfrac{1}{2}A = 20° 53'$
$\tfrac{1}{2}B = 27° 58'$
$\tfrac{1}{2}C = 41° 9'$
$A = [41° 46']$
$B = [55° 56']$
$C = [82° 18']$

Check:

$A + B + C = 180°.$

Exercises

Find the values of the angles in each of the following triangles:

1. $a = 173,$ $b = 98.6,$ $c = 230.$

2. $a = 8.067,$ $b = 1.765,$ $c = 6.490.$

3. $a = 1911,$ $b = 1776,$ $c = 1492.$

[1] Adding the four numbers above this line checks the subtractions $s - a$, $s - b$, etc.

Miscellaneous Problems

The instructor will select only a limited number of the following problems for actual computation by the student. The student should be required, however, to *outline* in writing the solution of a number of problems which he is not required actually to compute, and, when practicable, to block out a suitable check for each one of them.

1. From one corner P of a triangular field PQR the side PQ bears north 10° east 100 rods. QR bears north 63° east and PR bears north 38° 10′ east. Find the perimeter and area of the field.

2. The town B lies 15 miles east of A, C lies 10 miles south of A. X lies on the line BC, and the bearing of AX is south 46° 20′ east. Find the distances from X to the other three towns.

3. To find the length of a lake (Fig. 122), the angle $C = 48° 10'$, the side $a = 4382$ ft., and the angle $B = 62° 20'$ were measured. Find the length of the lake c, and check.

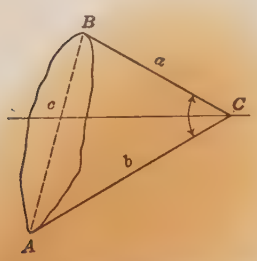


FIG. 122.—Diagram for Problem 3.

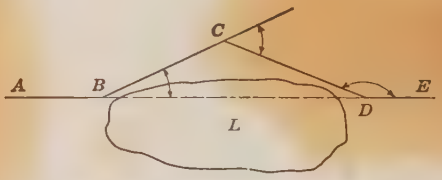


FIG. 123.—Diagram for Problem 4.

4. To continue a line past an obstacle L (Fig. 123) the line BC and the angles marked at B and C were measured and found to be 1842 ft., 28° 15′, and 67° 24′, respectively. Find the distance CD, and the angle at D necessary to continue the line AB; also compute the distance BD.

5. Find the longer diagonal of a parallelogram, two sides being 69.1 and 97.4 and the acute angle being 29° 34′.

What is the magnitude of the single force equivalent to two forces of 69.1 and 97.4 dynes respectively, making an angle of 29° 34′ with each other?

6. A force of 75.2 dynes acts at an angle of 35° with a force F. Their resultant is 125 dynes. What is the magnitude of F?

7 The equation of a circle is $\rho = 10 \cos \theta$. The points A and B on this circle have vectorial angles 31° and 54° respectively. Find the distance AB: (*a*) along the chord; (*b*) along the arc of the circle.

8. Find the lengths of the sides of the triangle enclosed by the straight lines:

$$\theta = 26°; \theta = 115°; \rho \cos (\theta - 45°) = 50.$$

9. A gravel heap has a rectangular base 100 ft. long and 30 ft. wide. The sides have a slope of 2 in 5. Find the number of cubic yards of gravel in the heap.

10. A point B is invisible and inaccessible from A and it is necessary to find its distance from A. To do this a straight line is run from A to P and continued to Q such that B is visible from P and Q. The following measurements are then taken: $AP = 2367$ ft.; $PQ = 2159$ ft.; $APB = 142° 37'.3$; $AQB = 76° 13'.8$. Find AB.

11. To determine the height of a mountain the angle of elevation of the top was taken at two stations on a level road and in a direct line with it, the one 5280 yd. nearer the mountain than the other. The angles of elevation were found to be 2° 45′ at the further station and 3° 20′ at the nearer station. Find the horizontal distance of the mountain top from the nearer station and the height of the mountain above it. Use S and T functions.

12. Explain how to find the distance between two mountain peaks M_1 and M_2: (*a*) when A and B at which measurements are taken are in the same vertical plane with M_1 and M_2; (*b*) when neither A nor B is in the same vertical plane with M_1 and M_2.

13. The sides of a triangular field are 534, 679 and 474 yd. The first bears north, and following the sides in the order here given the field is always to the left. Find the bearing of the other two sides and the area.

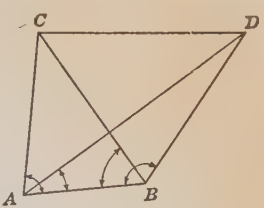


FIG. 124.—Diagram for Problem 16.

14. From a triangular field whose sides are 124, 96 and 104 rods a strip containing 10 acres is sold. The strip is of uniform width, having as one of its parallel sides the longest side of the field. Find the width of the strip.

15. Three circles are externally mutually tangent. Their radii are 5, 6, and 7 ft. Find the area and the perimeter of the three-cornered area enclosed by the circles and the length of a wire that will enclose the group of three circles when stretched about them.

16. To find the distance between two inaccessible objects C and D (Fig. 124), two points A and B are selected from which both objects

are visible. The distance AB is found to be 7572 ft. The following angles were then taken:

$ABD = 122° 37'$. $BAC = 80° 20'$.
$ABC = 70° 12'$. $BAD = 27° 13'$.

Find the distance DC and check.

17. A circle of radius a has its center at the point (ρ_1, θ_1). Find its equation in polar coordinates (use law of cosines).

18. A surveyor desired the distance of an inaccessible object O from A and B, but had no instruments to measure angles. He measured AA' in the line AO, BB' in the line BO; also AB, BA', and AB'. How did he find OA and OB?

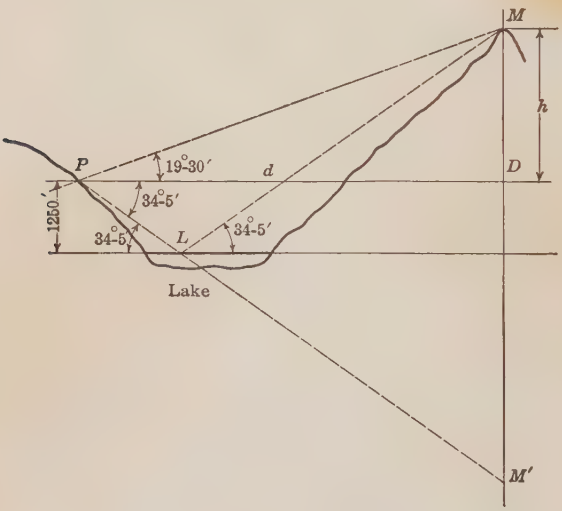


FIG. 125.—Diagram for Problem 20.

19. From a point A a distant object C bears north 32° 16′ west and from B the same object bears north 50° west. AB bears north 10° 39′ west. The distance AB is 1000 yd. Find the distance AC.

20. The angle of elevation of a mountain peak is observed to be 19° 30′. The angle of depression of its image reflected in a lake 1250 ft. below the observer is found to be 34° 5′. Find the height of the mountain above the observer and the horizontal distance to it (see Fig. 125).

21. One side of a mountain is a smooth eastern slope inclined at an angle of 26° 10′ to the horizontal. At a station A a vertical shaft is sunk to a depth of 300 ft. From the foot of the shaft two horizontal tunnels are dug, one bearing north 22° 30′ east, and the other south 65° east. These tunnels emerge at B and at C respectively. Find the lengths of the tunnels and the lengths of the sides of the triangle ABC.

22. A rectangular field $ABCD$ has side $AB = 40$ rods; $AD = 80$ rods. Locate a point P in the diagonal AC so that the perimeter of the triangle APB will be 160 rods. (*Hint:* Express perimeter as a function of angle at P.)

23. Find the area enclosed by the lines $y = x/2$, $y = \sqrt{3}\,x$, and the circle $x^2 - 10x + y^2 = 0$. (*Hint:* Change to polar coordinates.)

24. The displacement of a particle from a fixed point is given by

$$d = 2.5 \cos t + 2.5 \sin t.$$

What values of t give maximum and minimum displacements; what is the maximum displacement?

25. A quarter section of land is enclosed by a fence. A farmer wishes to make use of this fence and 60 rods of additional fencing in making a triangular field in one corner of the original tract. Find the field of greatest possible area. Show that it is also the field of maximum perimeter, under the conditions given.

26. A force $F_1 = 100$ dynes makes an angle of $\theta°$ with the horizontal, and a second force $F_2 = 50$ dynes makes an angle of 90° with F_1. Determine θ so that (*a*) the sum of the horizontal components of F_1 and F_2 shall be a maximum; (*b*) so that the sum of the vertical components shall be zero.

27. Find the area of the largest triangular field that can be enclosed by 200 rods of fence, if one side be 70 rods in length.

28. Change the equation of the curve $xy = 1$ to polar coordinates, rotate through $-45°$, and change back to rectangular coordinates.

29. A particle moves along a straight line so that the distance varies directly as $(\sin t + \cos t)$. When $t = \pi/4$, the distance is 10. Find the equation of motion.

30. From the top of a lighthouse 60 ft. high the angle of depression of a ship at anchor was observed to be 4° 52′; from the bottom of the lighthouse the angle was 4° 2′. Required the horizontal distance from the lighthouse to the ship and the height of the base of the lighthouse above the sea.

31. The line AB runs north and south. The line AC makes an angle of 52° 8′.6 with AB. Locate the line BC perpendicular to AB so that the area ABC shall be 1 acre.

32. A building casts a shadow 324 ft. long down the hillside on which it stands. The slope of the hillside is 15 ft. in 100 ft., and the elevation of the sun is 23° 27′. Find the height of the building.

33. To determine the distance of a fort A from a place B, a line BC and the angles ABC and BCA were measured and found to be 3225.5 yd., 60° 34′, and 56° 10′ respectively. Find the distance AB.

34. A balloon is directly over a straight level road, and between two points on the road from which it is observed. The points are 15,847 ft. apart, and the angles of elevation are 49° 12′ and 53° 29′. Find the height.

35. Two trees are on opposite sides of a pond. Denoting the trees by A and B, we measure $AC = 297.6$ ft.; $BC = 864.4$ ft., and the angle $ACB = 87° 43′$. Find AB.

36. Two mountains are 9 and 13 miles respectively from a town, and they include at the town an angle of 71° 36′. Find the distance between the mountains.

37. The sides of a triangular field are, in clockwise order, 534, 419, and 474 ft.; the first bears north; find the bearings of the other sides and the area.

38. Under what visual angle is an object 7 ft. long seen when the eye is 15 ft. from one end and 18 ft. from the other?

39. The shadow of a cloud at noon is cast on a spot 1600 ft. west of an observer, and the cloud bears south, 76° west, elevation 23°. Find its height.

CHAPTER X

SIMPLE HARMONIC MOTION AND WAVES

161. Trigonometric Functions and Waves.—It is necessarily but unfortunately true that most students, when they first study the trigonometric functions, obtain the impression that their principal application is to the solving of triangles. This impression arises from the fact that the student of trigonometry is usually not prepared to study other applications of these functions. The fact is, however, that the solution of triangles, while important, is but one of many applications, Indeed, the fundamental significance of the trigonometric functions arises from the fact that they are *periodic* functions, and are thus suited to the analytical description of any periodic phenomenon. A large portion of the important problems which scientists study are periodic in nature. Nearly all the problems of astronomy, and of the transmission of sound, light, and electricity, for example, are periodic, and their analysis requires a continual use of trigonometric functions. In most of these problems there is involved the notion of a "wave," whether it be a pressure "wave," such as constitutes sound; an electrodynamic "wave," such as constitutes light; or a wave on a body of water, such as a tidal wave. The connection between this notion of a wave and the notion of a simple harmonic motion, and the way in which the trigonometric functions are used in such problems will be explained, in its simplest aspects only, in this chapter.

162. Simple Harmonic Motion.—In Fig. 126, $x = OD = a \cos DOM$, where a is the radius of the circle. If now the point M be thought of as moving with constant or uniform speed on the circle, starting at A, or (which amounts to the same thing) if the radius OM be thought of as moving with constant angular

velocity, say k radians per second, starting from OA, then angle $DOM = kt$ and the position of the point D at time t is given by

$$x = a \cos kt, \tag{1}$$

where t is the time in seconds required for OM to move from position OA to position OM.

Let us study the motion of the point D as M moves on the circle with constant speed. D starts at A and moves to the left with increasing speed until it arrives at O, where its speed begins to decrease, decreasing to 0 at A'. Then the point moves to the right with increasing speed until it again passes through O, after which its speed diminishes, becoming 0 when it arrives at A. Then the whole motion is repeated. *A body whose position on a straight line is given at any instant by an equation of the form of Eq.* (1), *that is, one which moves as the point D does, is said to describe* **simple harmonic motion.** On account of the frequency with which this term will occur, we shall abbreviate it by the symbols S.H.M. Examples of bodies that move approximately in this way are: the bob of a pendulum, a point in the prong of a vibrating tuning fork, a point in a vibrating violin string, the particles of air during the passage of a sound wave.

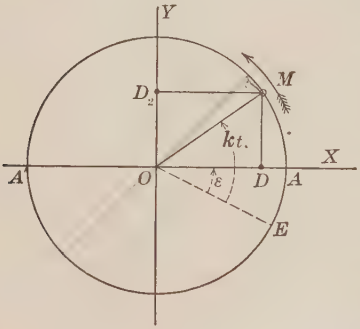


FIG. 126.

The motion is oscillatory in character and repeats itself in definite intervals of time. The length of this interval can be easily found by considering the motion of the point M on the circle. The point D, starting from any given position, will pass through this same position in the same direction after that interval of time which M requires to describe the circle, *i.e.*, after $2\pi/k$ sec., the time in which the radius OM describes the angle 2π radians at the rate of k radians per second. This time within which

a body executing S.H.M. performs a complete oscillation is called the **period** of the S.H.M. It is denoted by T. Thus

$$\mathbf{T = \frac{2\pi}{k}}. \tag{2}$$

This expression can be obtained directly from the equation $x = a \cos kt$ by means of the fact that the cosine is a periodic function of period 2π. The period T is the amount by which t must be increased in order to increase the angle kt by the amount 2π. If t be increased by the amount $2\pi/k$, then kt is increased by 2π, because

$$k\left(t + \frac{2\pi}{k}\right) = kt + 2\pi.$$

The number of complete periods per second is

$$\mathbf{N = \frac{1}{T} = \frac{k}{2\pi}}. \tag{3}$$

N is called the **frequency** of the S.H.M.

If, instead of counting time from the instant at which the auxiliary point M passed through A, we count it from the instant it passes through E, then $\angle EOM = kt$, and it is clear that $\angle AOM = (kt - \epsilon)$ if ϵ stands for the constant angle EOA. Then Eq. (1) becomes

$$x = a \cos (kt - \epsilon). \tag{4}$$

The number a is called the **amplitude,** ϵ is called the **epoch angle,** and $(kt - \epsilon)$ is called the **phase angle** of the S.H.M. represented by Eq. (4).

In like manner the point D_2, the projection of the point M upon the vertical diameter of the circle in Fig. 126 describes S.H.M. Its equation is

$$y = a \sin (kt - \epsilon), \tag{5}$$

where time t is measured from the instant M passes through E.

163. Mechanical Generation of S.H.M.—Figure 127 illustrates a way in which S.H.M. may be described by mechanical means. Let the uniformly rotating wheel OAB be provided with a pin M attached to its circumference and free to move in the slot of the cross-head as shown, the arm of the cross-head being restricted to vertical motion by suitable guides G_1G_1. Then, as the

wheel rotates, any point P of the arm of the cross-head describes S.H.M. in a vertical direction. The **amplitude** of the S.H.M. is the radius of the circle, or OB; its **period** is the time required for one complete revolution of the wheel.

If a tracing point be attached at P, bearing on a piece of smoked glass, HK, moving to the left with constant speed V, then the curve which the point P traces is a sinusoid. Let the abscissa x of the point P be measured from O', the point on the moving glass

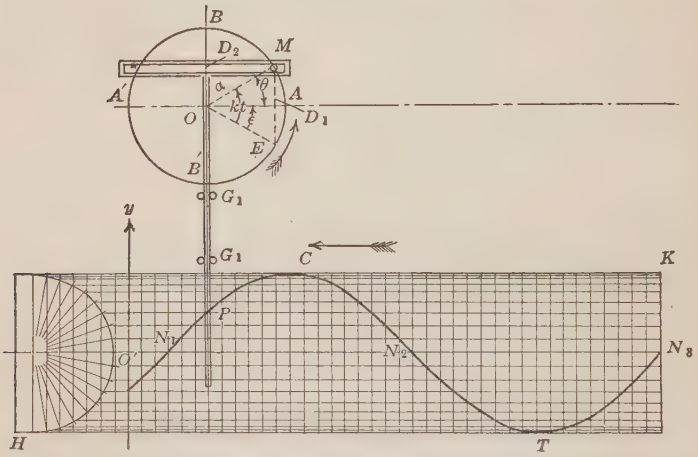


FIG. 127.—Mechanical generation of simple harmonic motion, and of a simple progressive wave.

which was exactly below O when M was at E; that is, at $t = 0$. Then, since O' moves to the left with velocity V,

$$x = Vt. \tag{1}$$

The ordinate y of the point P is, by Eq. (5) of the preceding section,

$$y = a \sin (kt - \epsilon). \tag{2}$$

Eliminating t between these two equations,

$$y = a \sin \left(\frac{kx}{V} - \epsilon \right), \tag{3}$$

the equation of a sinusoid.

Exercises

1. By what must the ordinates and the abscissas of the fundamental sinusoid $y = \sin x$ be multiplied to form the sinusoid $y = a \sin \frac{kx}{V}$? If the abscissas and the ordinates of a curve are multiplied by the same number, the curve is not distorted, but merely drawn to a different scale. What must the velocity V be in the above equation if the sinusoid Eq. (3) be undistorted? When V has this value, compare the velocity of the moving glass plate and the moving point M

2. Find the periods of the following S.H.M.:

(*a*) $y = 3 \sin 2t$.

(*b*) $y = 10 \sin (\frac{1}{2})t$.

(*c*) $y = 7 \cos 4t$.

(*d*) $y = a \sin 2\pi t$.

(*e*) $y = a \sin \left(10t - \frac{\pi}{3}\right)$.

(*f*) $y = a \sin \left(\frac{2t}{3} - \frac{2\pi}{5}\right)$.

(*g*) $y = a \sin (bt + c)$.

3. Give the amplitudes and the epoch angles in each of the instances given in Exercise 2.

4. The bob of a second's pendulum swings a maximum of 4 cm. each side of its lowest position. Considering the motion as rectilinear S.H.M., write the equation of motion.[1]

Write the equation of motion of a pendulum of the same length which was released from the end of its swing ½ sec. after the first pendulum was similarly released.

5. A particle moves in a straight line in such a way that its displacement from a fixed point of the line is given by $d = 2 \cos^2 t$. Show that the particle moves in S.H.M., and find the amplitude and period of the motion.

6. A particle moves in a vertical circle of radius 2 units with angular velocity of 20 radians per second. Counting time from the instant the particle was at its lowest position, write the equation of motion of its projection: (*a*) upon the vertical diameter; (*b*) upon the horizontal diameter; (*c*) upon the diameter bisecting the angle between the horizontal and the vertical

164. Composition of Two S.H.M.'s at Right Angles.—It is obvious that

$$x = a \cos kt$$

[1] The term *period* is used differently in the case of a pendulum than in the case of S.H.M. The time of a *swing* is the period of a pendulum; the time of a *swing-swang* is the period of a S.H.M.

represents a S.H.M. one-quarter of a period in advance of $x' = a \sin kt$, since $\sin\left(kt + \frac{\pi}{2}\right) = \cos kt$. A pair of S.H.M.'s possessing this property are said to be **in quadrature.** Equations (4) and (5) of Sec. **162** may be said to be in quadrature.

We have shown that if a point M, moving uniformly on a circle, be projected upon both the X- and Y-axes, two S.H.M.'s result. The phase angles of these two motions differ from each other by $\pi/2$. The converse of this fact, namely, that uniform motion in a circle may be the resultant of two S.H.M.'s in quadrature, is easily proved, for the two equations of S.H.M.

$$x = a \cos kt$$
$$y = a \sin kt$$

are obviously, the parametric equations of a circle. Hence the theorem:

Uniform motion in a circle may be regarded as the resultant of two S.H.M.'s of equal amplitudes and equal periods and differing by $\pi/2$ in phase angle.

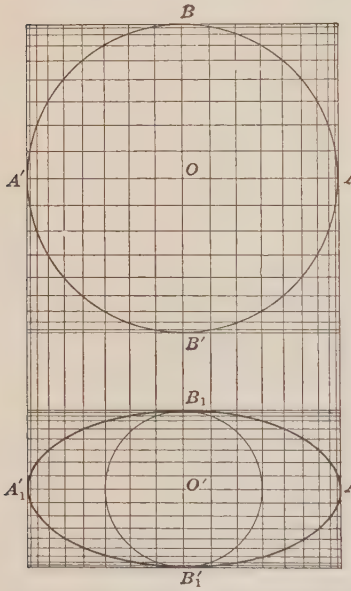


FIG. 128.—The circle and the ellipse considered as generated by two S.H.M.'s in quadrature.

This important truth is illustrated by Fig. 128. Let the X- and the Y-axes be divided proportionally to the trigonometric sine, as in Fig. 57. Through the points of division of the two axes draw lines perpendicular to the axes, thus dividing the plane into a large number of small rectangles. Starting at the end of one of the axes, and sketching the diagonals of successive cornering rectangles, the circle $ABA'B'$ is drawn.

If the same construction be carried out for the case in which the Y-axis is divided proportionally to $b \sin kt$ and in which the X-axis is divided proportionally to $a \sin kt$, the ellipse $A_1B_1A'_1B'_1$ results. These facts are merely a repetition of the statements made in Sec. **79.**

165. Waves.—Let Fig. 129 represent a section obtained by passing at any instant a vertical plane perpendicular to the crests of a series of small waves on the surface of a body of water. The wavy line represents the appearance of the surface at any instant. It is a fact that its equation is, in the case of small waves or ripples,

$$y = a \sin hx. \tag{1}$$

This is the undistorted sinusoid $y = \sin x$ with its ordinates multiplied by a and its abscissas by $1/h$. With proper choice of a and h, Eq. (1) can be made to represent a ripple form of any amplitude and of any distance between successive crests. Figure 129 represents the section of the surface at any instance, say, $t = 0$. Now such waves move forward with a constant velocity, which

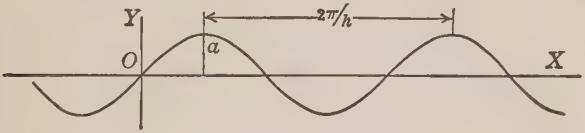


Fig. 129.

we shall call V. The wavy form is sinusoidal in section, but, of course, it is not fixed, but keeps moving ahead. We have learned that a curve may be translated to the right a distance b by substituting $x - b$ for x. Thus the substitution of $x - Vt$ for x translates a curve to the right a distance Vt which increases uniformly with time. Hence, if in the equation of a curve we substitute $x - Vt$ for x, the result is the equation of the original curve moving to the right with velocity V. The desired equation of the moving wave is thus

$$y = a \sin h(x - Vt). \tag{2}$$

The product hV may be represented by a single letter k, and the equation rewritten

$$\mathbf{y = a \sin (hx - kt).} \tag{3}$$

Because of the presence of the variable t, Eq. (3) is not the equation of a fixed *sinusoid*, but of a *moving sinusoid* or **wave.**

Applying the same terms used for S.H.M., the expression $hx - kt$ is the **phase angle,** the expression $+kt$ is the **epoch angle** and a is the **amplitude** of the wave (see Fig. 130*a* and *c*).

The expression $hx - kt$ is a *linear* function of the variables x and t. The sine or cosine of this function is called a **simple harmonic function** of x and t.

The *wave form* on the surface of water moves along with fixed velocity V. The particles of water, however, do not share in this

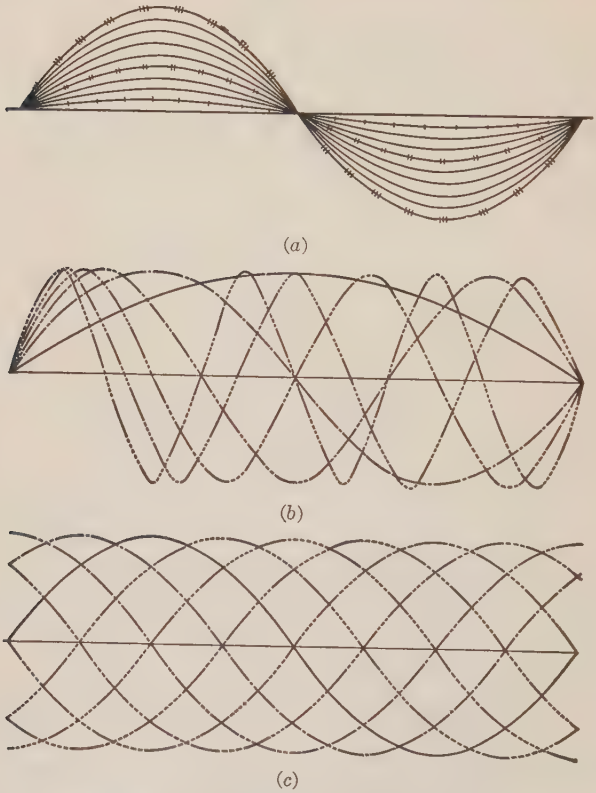


FIG. 130.—Wave forms, (*a*) of different amplitude; (*b*) of different wave lengths; (*c*) of different phase or epoch angles.

forward motion. Each particle on the surface moves *up and down* in a vertical line as a wave form passes it. In fact, we shall now see that each particle describes S.H.M. in a vertical direction.

To examine the motion of a single particle of water, we have only to regard x as constant, say $x = x_1$, in Eq. (3) above. The displacement of this particle is then given by

$$y = a \sin (hx_1 - kt)$$

or

$$y = -a \sin (kt - hx_1),$$

that is,

$$y = a \sin (kt - hx_1 - \pi). \tag{4}$$

This is the equation of a S.H.M. whose period is $T = 2\pi/k$. The epoch angle is $hx_1 + \pi$. This will be different for different particles. This means that the phase angles of the S.H.M. of successive particles differ, but they all oscillate up and down with the same period $2\pi/k$.

166. Wave Length, and Period or Periodic Time.—The wave length L of the progressive wave (Eq. (3), Sec. **165**) is defined to be the distance from crest to crest (or from trough to trough), and is clearly equal to the distance from crest to crest on the corresponding stationary sinusoid $y = a \sin hx$. But this latter curve is made from the curve $y = a \sin x$, whose wave length is 2π, by multiplication of the abscissas by $1/h$. Hence the wave length of $y = a \sin hx$ or of $y = a \sin (hx - kt)$ is given by

$$\mathbf{L = \frac{2\pi}{h}}. \tag{1}$$

The period or periodic time T of the wave Eq. 3, Sec. **165** is defined as the time required for the wave to progress a distance equal to one wave length. Thus $L = VT$, or

$$\mathbf{T = \frac{L}{V} = \frac{2\pi}{hV} = \frac{2\pi}{k}}. \tag{2}$$

To contrast **wave length** and **period,** think of a person in a boat anchored at a fixed point in a lake. The time that the person must wait *at that fixed point* (x constant) for crest to follow crest is the *periodic time.* The *wave length* is the distance he observes between crests *at a given instant of time* (t constant).

The number of periods per unit of time is called the **frequency** of the wave. Hence, if N represent the frequency of the wave,

$$\mathbf{N = \frac{1}{T} = \frac{k}{2\pi}}. \tag{3}$$

167. L and T Equation of a Wave.—If we solve Eqs. (1) and (2), Sec. **166,** for h and k respectively, and substitute these values of h and k in the equation

$$y = a \sin (hx - kt),$$

we obtain

$$\mathbf{y = a \sin 2\pi\left[\frac{x}{L} - \frac{t}{T}\right].} \tag{1}$$

From this form it is seen that the argument of the sine increases by 2π when either x increases by an amount L or when t increases by the amount T. By use of Eq. (2), Sec. **166,** the last equation may also be written

$$\mathbf{y = a \sin \frac{2\pi}{L}(x - Vt).} \tag{2}$$

168. Lead.—Consider the two waves,

$$y = a \sin \frac{2\pi}{L}(x - Vt) \tag{1}$$

$$y = a \sin \frac{2\pi}{L}(x - Vt - E). \tag{2}$$

The amplitudes, the wave lengths, and the velocities are the same in each, but the second wave is in advance of the first by the amount E (measured in linear units), for the second equation can be obtained from the first by substituting $x - E$ for x, which translates the curve the amount E in the OX direction. In this case E is called the **lead** (or the **lag** if negative) of the second wave compared with the first. The lead is a *linear* magnitude measured in centimeters, inches, feet, etc.

The tidal wave in midocean, the ripples on a water surface, the wave sent along a rope that is rapidly shaken by the hand, are illustrations of progressive waves of the type discussed above. Sound waves also belong to this class if the alternate condensations and rarefactions of the medium be graphically represented by ordinates. The ordinary progressive waves observed upon a lake or the sea are not, however, progressive waves of this type. The surface of the water in this case is not sinusoidal in form, but is represented by another class of curves known in mathematics as *trochoids.*

Exercises

1. Derive the amplitude, the wave length, the periodic time, and the velocity of propagation of the following waves:

(*a*) $y = a \sin (2x - 3t)$.

(*b*) $y = 5 \sin (0.75x - 1000t)$.

(*c*) $y = 10 \sin \left(\frac{x}{2} - \frac{t}{3}\right)$.

(*d*) $y = 50 \sin \frac{2\pi}{7}(x - 3t)$.

(*e*) $y = 100 \sin \frac{2\pi}{25}(x - 20t - 4)$.

(*f*) $y = 100 \sin (5x + 4t)$.

(*g*) $y = 0.025 \sin \frac{2\pi}{4}\left(\frac{x + t}{3}\right)$.

2. Write the equation of a progressive sinusoidal wave whose height is 5 ft., length 40 ft. and velocity 4 miles per hour.

3. Write the equation of a wave of wave length 10 m., height 1 m. and velocity of propagation 3.5 miles per hour. (NOTE: 1 mile = 1.609 km.)

4. Sound waves of all wave lengths travel in still air at 70°F. with a velocity of 1130 ft. per second. Find the wave length of sound waves of frequencies 256, 128, and 600 per second.

169. Stationary Waves.—The form of a violin string during its free vibration is sinusoidal, but the nodes, crests, troughs, etc. are stationary and not progressive as in the case of the waves

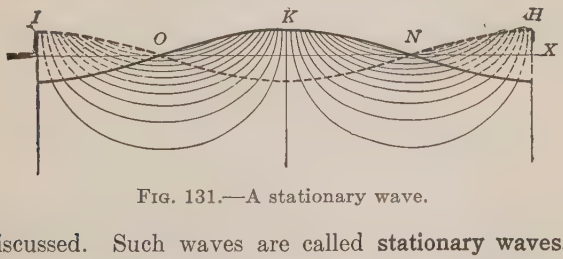


FIG. 131.—A stationary wave.

just discussed. Such waves are called **stationary waves.** The water in a basin or even in a large pond or lake is also capable of vibrating in this way. Figure 131 may be used to illustrate the stationary waves of this type, either of a musical string or of the water surface of a lake, but in the case of a vibrating string the ends must be supposed to be fastened at the points *O* and *N*. The shores of the lake may be taken at *I* and *K* or at *I* and *H*, etc. As is well known, such bodies are capable of vibrating in segments, so that the number of nodes may be large, as is the case, for example, when a violin string is sounding a "harmonic." A stationary

wave on the surface of a lake or pond is known as a **seiche,** and was first noted and studied on Lake Geneva, Switzerland. The amplitudes of seiches are usually small, and must be studied by means of recording instruments so set up that the influence of progressive waves is eliminated. The maximum seiche recorded on Lake Geneva was about 6 ft., although the ordinary amplitude is only a few centimeters.

The waves which pass down a violin string are reflected at the end and return, so that there are, at any instant, waves moving along the string in both directions. Stationary waves may always be considered the result of oppositely traveling progressive waves, and in most instances, as in the case just cited, reflections play a vital rôle in maintaining the system of waves. The equation of a stationary wave may be found by adding the ordinates of a progressive wave

$$y = a \sin (hx - kt) \tag{1}$$

traveling to the right ($k > 0$) to the ordinates of a progressive wave

$$y = a \sin (hx + kt) \tag{2}$$

traveling to the left.

Expanding the right members of Eqs. (1) and (2) by the addition formula for the sine and adding

$$\mathbf{y = 2a \cos kt \sin hx,} \tag{3}$$

or in terms of L and T (Sec. **166,** Eqs. (1) and (2)),

$$\mathbf{y = 2a \cos\left(\frac{2\pi t}{T}\right) \sin\left(\frac{2\pi x}{L}\right).} \tag{4}$$

In Fig. 131, the origin is at O and the X-axis is the line of nodes ONX. If in Eq. (3) we look upon $2a \cos kt$ as the variable amplitude of the sinusoid

$$y = \sin hx,$$

we note that the nodes of the sinusoid remain stationary but that the amplitude $2a \cos kt$ changes as time goes on. When $t = 0$, the sine curve has amplitude $2a$ and wave length $2\pi/h$. When $t = \pi/2k$, or $T/4$, the sinusoid is reduced to the straight line $y = 0$. When $t = \pi/k$, or $T/2$ the curve is the sinusoid

$$y = -2a \sin hx,$$

which has a trough where the initial form had a crest, or *vice versa*.

Exercises

In the following exercises the height of the wave means the maximum rise above the line of nodes. When a seiche is uninodal, the shores of the lake correspond to the points I and K (Fig. 131). When a seiche is binodal, the points I and H are at the lake shore.

1. From the equation of a stationary wave in the form $y = 2a \sin 2\pi x/L \cos 2\pi t/T$, show that K (Fig. 131) is at its lowest depth for $t = T/2, 3T/2, 5T/2, \ldots$

2. Henry observed a 15-hr. uninodal seiche in Lake Erie, which was 396 km. in length. Write the equation of the principal or uninodal stationary wave if the amplitude of the seiche was 15 cm.

3. A small pond 111 m. in length was observed by Endros to have a uninodal seiche of period 14 sec. Write the equation of the stationary wave if the amplitude be a.

4. Forel reports that the uninodal longitudinal seiche of Lake Geneva has a period of 73 min. and that the binodal seiche has a period of 35½ min. The transverse seiche has a period of 10 min. for the uninodal and 5 min. for the binodal. The longitudinal and transverse axes of the lake are 45 and 5 miles respectively. Write the equation of these different seiches.

5. A standing wave or uninodal seiche exists on Lake Mendota of period 22 min. If the maximum height be 8 in. and the distance across the lake be 6 miles, write the equation of the seiche.

170*. Compound Harmonic Motion and Compound Waves.— The addition of two or more simple harmonic functions of frequencies which are multiples of the frequency of a given first or **fundamental harmonic** gives rise to **compound harmonic motion.** Thus,

$$y = a \sin kt + b \sin 3kt$$

corresponds to the superposition of a S.H.M. of period $2\pi/3k$ and amplitude b upon a fundamental S.H.M. of period $2\pi/k$ and amplitude a. To compound motions of this type, there correspond compound waves of various sorts, such as a fundamental sound wave with overtones, or tidal waves in restricted bays or harbors. The graphs of the curves

$$y = \sin x + \sin 2x$$
$$y = \sin x + \sin 3x$$

are easily constructed. They may be drawn by adding the ordinates of the various sinusoids constructed on the same axis, as in Fig. 132.

To compound the curves, first draw the component curves, say, $y = \sin x$ and $y = \sin 3x$ of Fig. 132. Then use the edge of a piece of

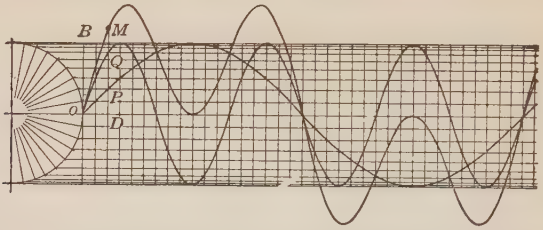


FIG. 132.—The curves $y = \sin x$, $y = \sin 3x$ and the compound curve $y = \sin x + \sin 3x$.

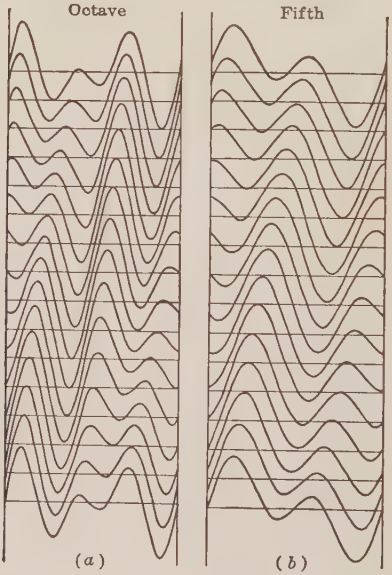


FIG. 133.—The curves (*a*) $y = \sin x + \sin\left(2x + \frac{2\pi n}{16}\right)$ and (*b*) $y = \sin 2x + \sin\left(3x + \frac{2\pi n}{16}\right)$, for $n = 0, 1, 2, \ldots 15$. (*From Thomson and Tait.*)

paper divided proportionally to $\sin x$ (that is, like the scale OB (Fig. 132)) and use this as a scale by means of which the successive

ordinates of a given x may be added. For example, to locate the point on the composite curve corresponding to the abscissa OD (Fig. 132) we must add DP and DQ. Hence place vertically at P the lower end of the paper scale just mentioned. The sixth scale division above P on this scale will then locate the required point M of the composite scale.[1]

In Fig. 133 the curves:

$$y = \sin x + \sin \frac{(2x + 2\pi n)}{16}$$

$$y = \sin 2x + \sin \frac{(3x + 2\pi n)}{16}$$

are shown for values of $n = 0, 1, 2, \ldots, 15$ in succession—that is, for successive phase differences corresponding to one-sixteenth of the wave length of the fundamental $y = \sin x$.

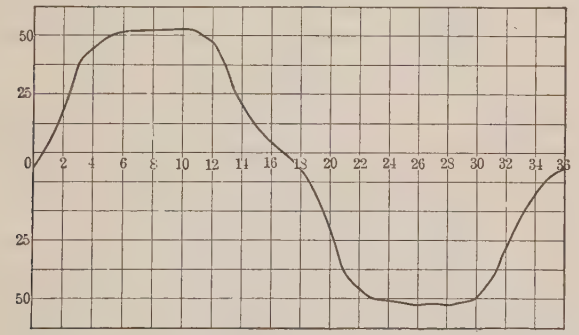


FIG. 134.—An alternating current curve. Only odd harmonics are present.

Wave forms compounded from the odd harmonics only are especially important, as alternating-current curves are of this type (see Fig. 134).

171*. Harmonic Analysis.—Fourier showed in 1822 in his "Analytical Theory of Heat" that a periodic single-valued function, say, $y = f(x)$, under certain conditions of continuity, can be represented by the sum of a series of sines and cosines of the multiple angles of the form

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_2 \cos 3x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

[1] Note that if the method described be used, there is really no need of drawing the curve $y = \sin 3x$. If both curves are drawn, ordinates may conveniently be added with bow dividers.

This means, for example, that it is always possible to represent the comples tidal wave in a harbor, by means of the sum of a number of simple waves or harmonics. The term **harmonic analysis** is given to the process of determining these sinusoidal components of a compound periodic curve. In Sec. **170** we have performed the direct operation of finding the compound curve when the component harmonics are given. The inverse operation of finding the components when the compound curve is given is much more difficult, and its discussion must be postponed to a later course.

172*. Connecting-rod Motion.—If one end of a straight line B be required to move on a circle while the other end of the line A moves on a straight line passing through the center of the circle, the resulting motion is known as **connecting-rod motion.** The connecting rod

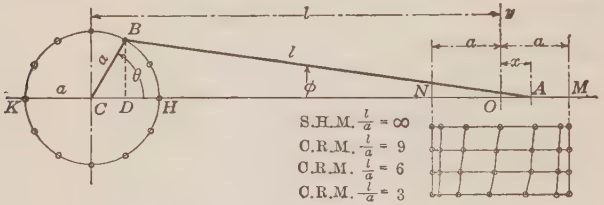


FIG. 135.—Connecting rod motion.

of a steam engine has this motion, as the end attached to the crank travels in a circle while the end attached to the cross-head travels in a straight line. The motion of the head A (Fig. 135) of the connecting rod is approximately S.H.M. The approximation is very close if the connecting rod be very long in comparison with the diameter of the circle.

A second approximation to the motion of the point A can be obtained by introducing the second harmonic or octave of the fundamental. In Fig. 135, let the radius of the circle be a and the length of the connecting rod be l. The length of the stroke MN is $2a$, and the origin may conveniently be taken at the mid-point of the stroke, O. When B is at H, A is at M and when B is at K, A is at N. Then $MH = NK = l$ and $OC = l$. Now

$$x = CA - CO = CA - l = CD + DA - l \tag{1}$$

But

$$CD = a \cos \theta \tag{2}$$

and

$$DA = \sqrt{l^2 - BD^2} = \sqrt{l^2 - a^2 \sin^2 \theta}. \tag{3}$$

Hence,

$$x = a\cos\theta + l\sqrt{1 - \left(\frac{a^2}{l^2}\right)\sin^2\theta} - l. \tag{4}$$

Approximating the radical by Sec. **105** ($\sqrt{1-x} = 1 - x/2$) we obtain

$$x = a\cos\theta + l\left(1 - \frac{a^2\sin^2\theta}{2l^2}\right) - l. \tag{5}$$

Since $\sin^2\theta = (1 - \cos 2\theta)/2$, we obtain

$$x = a\cos\theta + \frac{a^2}{4l}\cos 2\theta - \frac{a^2}{4l}, \tag{6}$$

which is approximately true as long as l is much greater than a.

It is seen from the above result that the second approximation to connecting-rod motion contains as overtone the *octave*, or *second* harmonic, $a^2/4l \cos 2\theta$, in addition to the first or fundamental harmonic $a\cos\theta$.

Exercises

1. Draw the curve corresponding to Eq. (5) above if $a = 1.15$ in., and $l = 3$ in.

2. The motion of a slide valve is given by an equation of the form

$$y = a_1 \sin(\theta + \epsilon) + a_2 \sin(2\theta + 90°).$$

Draw the curve if $a_1 = 100$, $a_2 = 25$, $\epsilon = 40°$, using θ as the abscissa in rectangular coordinates.

CHAPTER XI

COMPLEX NUMBERS

173. Scales of Numbers.—To measure any magnitude, we apply a unit of measure and then express the result in terms of numbers. Thus, to measure the volume of the liquid in a cask we may draw off the liquid a measure full at a time, in a gallon measure, and conclude for example that the number of gallons is $12\frac{1}{2}$. In this case the number $12\frac{1}{2}$ is taken from the *arithmetical scale* of numbers, 0, 1, 2, 3, 4, . . . If we desire to measure the height of a stake above the ground, we may apply a foot rule and say, for example, that the height in inches above the ground is $12\frac{1}{2}$, or, if the positive sign indicates height *above* the ground, we may say that the height in inches is $+12\frac{1}{2}$. In this latter case the number $+12\frac{1}{2}$ has been selected from the *algebraic scale of* numbers . . . -4, -3, -2, -1, 0, $+1$, $+2$, $+3$, $+4$, . . .

The scale of numbers which must be used to express the value of a magnitude depends entirely upon the nature of the magnitude. The attempt to express certain magnitudes by means of numbers taken from the algebraic scale may sometimes lead, as every student of algebra knows, to meaningless absurdities. Thus a problem involving the number of sheep in a pen, or the number of marbles in a box, or the number of gallons in a cask cannot lead to a *negative* result, for the magnitudes just named are *arithmetical quantities* and their measurement leads to a number taken from the *arithmetical scale.* The absurdity that sometimes appears in results to problems concerning these magnitudes is due to the fact that one attempts to apply the notion of algebraic number to a magnitude that does not permit of it. Science deals with a great many different kinds of magnitudes, the measurement of some of which leads to arithmetical numbers while the measurement of others leads to algebraic numbers; the remarkable fact is that two different number scales serve adequately to express magnitudes of so many different sorts. The magnitudes of science

are so various in kind that one might reasonably expect that the variety of number systems required in the mathematics of these sciences would be very great.

The arithmetical scale is used when we enumerate the number of gallons in a cask and say: 0, 1, 2, 3, . . . If we observe three gallons in the cask, and then remove one, we note those remaining and say *two;* we may remove another gallon and say *one;* we may remove the last gallon and say *zero;* but now the magnitude has come to an end. The algebraic scale is used when we measure in inches the height of a stake above the ground and say *three.* We may drive the stake down an inch and say *two;* we may drive the stake another inch and say *one;* we may drive the stake another inch and say *zero,* or "level with the ground"; but, unlike the case of the gallons in the cask, we need not stop but may drive the stake another inch and say *one below the ground,* or, for brevity, *minus one;* and so on.

Many of the magnitudes considered in science are completely expressed by means of arithmetical numbers only; for example, such magnitudes as *density* or specific gravity; *temperature;*[1] *electrical resistance; quantity of energy,* such as ergs, joules, or foot-pounds; *power,* such as horsepower, kilowatts, etc. All of the magnitudes just mentioned are **scalar,** that is, they exist in one sense only—not in one sense and also in the opposite sense, as do forces, velocities, distances, as explained above. The arithmetical scale of numbers is, therefore, ample, for their expression.

The distinction, then, between an algebraic number and an arithmetical number is the notion of *sense* which must always be associated with any algebraic number. Thus an algebraic number not only answers the question "*how many*" but also affirms the *sense* in which that number is to be understood; thus the algebraic number $+12\frac{1}{2}$, if arising in the measurement of angular magnitude, refers to an angular magnitude of $12\frac{1}{2}$ units (degrees, or radians, etc.) taken in the sense defined as *positive rotation.*

174. Algebraic Number Not the Most General Sort.—Algebraic numbers, although more general than arithmetical numbers, are themselves quite restricted, for, each algebraic number corresponds to a point of the algebraic scale. But for points not on the scale there corresponds no algebraic number, that is, the algebraic scale is one-dimensional. It is thus seen that there is an

[1] Temperature is an arithmetical quantity, since there is an *absolute zero* of temperature. Temperature does not exist in *two opposite senses,* but in a *single sense.*

opportunity of enlarging our conception of number if we can remove the restriction of one dimension—that is, if we can get out of the line of the algebraic scale and set up a number system such that one number of the system will correspond, for example, to *each point of a plane*, and such that one point of the plane will correspond to each number of the system. We will seek, therefore, an extension or generalization of the number system of algebra that will enable us to consider, along with the points of the algebraic scale, those points which lie without it.

175. Numbers as Operators.—The extension of the number system mentioned in the last section may be facilitated by changing the conception usually associated with symbols of number. The usual distinction in algebra is between *symbols of number* and *symbols of operation*. Thus a symbol which may be looked upon as answering the question "how many" is called a number, while a symbol which tells us to do something is called a **symbol of operation,** or, simply, an operator. Thus in the expression $\sqrt{2}$, $\sqrt{}$ is a symbol of operation and 2 is a number. A symbol of operation may always be read as a verb in the imperative mood; thus we may read $\sqrt{x}$: "Take the square root of x." Likewise, $\log x$, and $\cos \theta$ may be read: "Find the logarithm of x," "Take the cosine of θ." In these expressions "log" and "cos" are symbols of operation; they tell us to do something; they do not answer the question "how many" or "how much," and hence are not numbers. Here we speak of $\sqrt{}$, log, cos, as **operators;** we speak of x as the **operand,** or that which is operated upon.

It is interesting to note that *any number may be regarded as a symbol of operation;* by doing so we very greatly enlarge some original conceptions. Thus, 10 may be regarded not only as *ten,* answering the question "how many," but it may quite as well be regarded as denoting the *operation* of taking unity, or any other operand that follows it, ten times; to express this we may write $10 \cdot 1$, in which 10 may be called a **tensor** (that is, "stretcher"), or a symbol of the operation of stretching a unit until the result obtained is tenfold the size of the unit itself. In the same way the symbol 2 may be looked upon as denoting the operation of *doubling* unity, or of doubling any operand that follows it; likewise, the tensor 3 may be looked upon as a *trebler*, 4 as a *quadrupler*, etc.

With the usual understanding that any symbol of operation operates upon that which follows it, we may write compound operators like $2 \cdot 2 \cdot 3 \cdot 1$. Here 3 denotes a trebler and $3 \cdot 1$ denotes that the unit is to be trebled, 2 denotes that this result is to be doubled and the next 2 denotes that *this* result is to be doubled. Thus, representing the unit by a line running to the right, we have the following representation of the operators:

The unit $\rightarrow$

$3 \cdot 1$ $\rightarrow\!\!\rightarrow\!\!\rightarrow$

$2 \cdot 3 \cdot 1$ ——→——→——→

$2 \cdot 2 \cdot 3 \cdot 1$ ——————→——————→——————→

Notice the significance that should now be assigned to an exponent attached to these (or other) symbols of operation. The exponent means *to repeat the operation designated by the operator;* that is, the operation designated by the base is to be performed, and performed again on *this* result, and so on, the number of operations being denoted by the exponent. Thus, 10^2 means to perform the operation of repeating unity 10 times (indicated by 10) and then to perform the operation of repeating the result 10 times, that is, it means $10(10 \cdot 1)$. Also, 10^3 means $10[10(10 \cdot 1)]$. Likewise, $\log^2 30$ means log (log 30) which, if the base be 10, equals log 1.4771, or, finally, 0.1694. An apparent exception occurs in the case of the trigonometric functions. The expression $\cos^2 x$ *should* mean, in this notation, cos (cos x) but, because trigonometry is historically so much older than the ideas here expressed, the expression $\cos^2 x$ came to be used for $(\cos x)^2$, or $(\cos x) \times (\cos x)$, but $\cos^{-1} \theta$ **means arc cos** θ, *not* $1/\cos\theta$.

To be consistent with the notation of elementary mathematics, the expression $\sqrt{4}$, looked upon as a symbol of operation, must denote an operation which must be performed *twice* in order to be equivalent to the operation of quadrupling; that is, such that $(\sqrt{4})^2 = 4$. Likewise, $\sqrt[3]{4}$ denotes an operation which must be performed three times in succession in order to be equivalent to quadrupling. But we know that the operation denoted by 2, if performed twice, is equivalent to quadrupling; therefore, $\sqrt{4} = 2$, etc. Just as 4^2, 4^3, etc., may be called *stronger* tensors than a single 4, so $\sqrt{4}$, $\sqrt[3]{4}$, etc. may be called *weaker* tensors than the operator 4.

176. Reversor.—The expression (-1), looked upon as a symbol of operation, is not a tensor, as it leaves the size unchanged of that upon which it operates. If this operator be applied to any magnitude, it will change the sense in which the magnitude is then taken to exactly the opposite sense. Thus, if 6 stands for 6 hr. *after*, then $(-1)(6)$ stands for 6 hr. *before* a certain event, and (-1) is the symbol of this operation of *reversing the sense of the magnitude.* Also if 6 stands for a line running 6 units to the *right* of a certain point, then $(-1)(6)$ stands for a line running 6 units to the *left* of that point; so that (-1) is the symbol which denotes the operation of turning the straight line through 180°. As 2, 3, 4, when looked upon as symbols of operations, were called tensors, the operator (-1) may conveniently be designated a **reversor.**

Exercises

Show graphically the effect of the operations indicated in each of the following exercises. Take as the initial unit operand a straight line ½ in. long extending to the right of the zero or initial point. Explain each expression as consisting of the operand unity and symbols of operation—tensors, reversors, etc., which operate upon it, one after the other, in a definite order.

1. $2 \cdot 3 \cdot 1$.
2. $3 \cdot 3 \cdot 1$.
3. $-1 \cdot 3 \cdot 1$.
4. $2^3 \cdot 1$.
5. $\sqrt{3} \cdot 1$.
6. $(\sqrt[3]{2})^2 \cdot 1$.
7. $\sqrt{9} \cdot \sqrt{4} \cdot 1$.
8. $(\sqrt{4})^3 \cdot (-1) \cdot 1$.
9. $(-1)^3 \cdot 2^2 \cdot 3 \cdot 1$.
10. $3 \cdot 3^2 \cdot 1$.
11. $(-1)^2 \cdot 2^4 \cdot 1$.
12. $3.(-1) \cdot \sqrt{2} \cdot 1$.
13. $(\sqrt{2}) \cdot (-1)^{100} \cdot 1$.
14. $\sqrt[5]{10} \cdot 2 \cdot (-1) \cdot 1$.

15. A tensor, if permitted to operate 7 times in succession, will just double the operand Symbolize this tensor.

16. A tensor, if permitted to operate 5 times in succession, will quadruple the operand. Symbolize this tensor.

177. Versors.—The expression $\sqrt{-1}$ cannot consistently, with the meaning already assigned to $\sqrt{}$ and (-1), be looked upon as answering the question "how many," and therefore is not a number in that sense; yet if we consider $\sqrt{-1}$ as a *symbol of operation*, it can be given a meaning consistent with the operators already considered, for if 2 is the operator that doubles, and

$\sqrt{2}$ is the operator that *when used twice* doubles, then, since (-1) is the operator that reverses, the expression $\sqrt{-1}$ should be an operator which, when used twice, reverses. So, as (-1) may be defined as the symbol which operates to turn a straight line through an angle of 180°, in a similar way *we may define the expression* $\sqrt{-1}$ *as a symbol which denotes the operation of turning a straight line through an angle of* 90° *in the positive direction.* The restriction of positive rotation is inserted to make the definition unique.

The symbols (-1) and $\sqrt{-1}$ are not tensors. They do not represent a *stretching* or *contracting* of the operand. Their effect is merely to *turn* the operand to a new direction; hence these symbols may be called **versors,** or "turners."

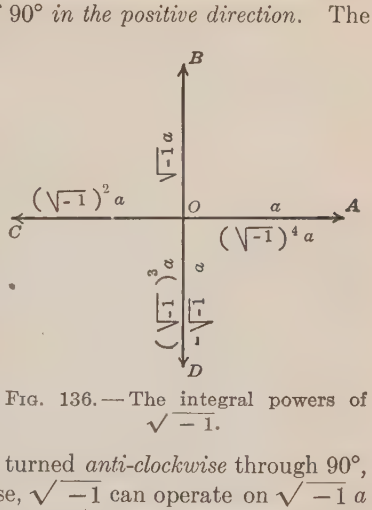


FIG. 136.—The integral powers of $\sqrt{-1}$.

178. The Operator $\sqrt{-1}$. In Fig. 136 let a be any line. Then a operated upon by $\sqrt{-1}$ (that is, $\sqrt{-1}\,a$) is a turned *anti-clockwise* through 90°, which gives OB. Now, of course, $\sqrt{-1}$ can operate on $\sqrt{-1}\,a$ just as well as on a. Then $\sqrt{-1}\,(\sqrt{-1}\,a)$, or OC, is $\sqrt{-1}\,a$ turned positively through 90°. Similarly,

$$\sqrt{-1}\,[\sqrt{-1}\,(\sqrt{-1}\,a)] \text{ is } \sqrt{-1}\,(\sqrt{-1}\,a)$$

turned through 90°, etc.

As we are at liberty to consider two turns of 90° as equivalent to one turn of 180°, therefore, $\sqrt{-1}\,(\sqrt{-1}\,a) = (-1)\,a$. Now $OD = (-1)OB$, $OD = (-1)(\sqrt{-1}\,a)$; but also $OD = \sqrt{-1}\,(-a)$, therefore, $(-1)\sqrt{-1}\,a = \sqrt{-1}\,(-a)$. Thus the student may show many like relations.

The operator $\sqrt{-1}$ is usually represented by the symbol i and will generally be so represented in what follows.

Exercises

Interpret each of the following expressions as a symbol of operation:

1. 2, 3, 4, -1.

2. 3^2. 2^3, 4^0, $(-1)^2$, $(-1)^5$.

3. $\sqrt{2}, \sqrt{3}, \sqrt{-1}, \sqrt[3]{2}, \sqrt[3]{-1}$.

Select a convenient unit and construct each of the following expressions geometrically, explaining the meaning of each operator:

4. $2 \cdot 3 \cdot 5 \cdot 1$.
5. $2^3 \cdot (-1) \cdot 1$.
6. $3 \cdot \sqrt{-1} \cdot 2 \cdot 1$.
7. $(-1)^2 \cdot \sqrt{-1} \cdot 1$.
8. $2^2 \cdot (-1)^3 \cdot (\sqrt{-1})^0 \cdot 1$.
9. $3 \cdot \sqrt{-1} \cdot (-1) \cdot \sqrt{-1} \cdot 1$.

179. Complex Numbers.—An expression of the form $a + bi$ is called a **complex number,** since it contains a term taken from each of the following scales, so that the unit is not single but double or complex:

$$\ldots -3, -2, -1, 0, +1, +2, +3, \ldots$$
$$\ldots -3i, -2i -i, 0, +i, +2i, +3i, \ldots$$

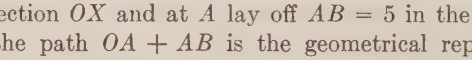
Any number belonging to the first scale is called a **real number,** any number belonging to the second scale is called a **pure imaginary.**

It is important to note that the only element common to the two series in this complex scale is 0.

The explanation of the meaning of the symbol $(a + bi)$ will be given in the following section. It will be shown in subsequent theorems that any expression made up of the sum, product, power or quotient of complex numbers may be put in the form $a + bi$, in which both a and b are real.

180. Meaning of a Complex Number.—Any real number, or any expression containing only real numbers, may be considered as locating a point in a *line.*

Thus, suppose we wish to draw the expression $2 + 5$. Let O be the zero point and OX the positive direction. Lay off $OA = 2$ in the direction OX and at A lay off $AB = 5$ in the direction OX. Then the path $OA + AB$ is the geometrical representation of $2 + 5$.



Any complex number may be taken as the representation of the position of a point in a plane. For, suppose $c + di$ is the complex number. Let O (Fig. 137) be the zero point and OX the positive direction. Lay off $OA = +c$ in the direction OX and at A erect

di in the direction OY, instead of in the direction OX as in the last example. It is agreed to consider the step to the right, OA, followed by the step upward, AP, as the *meaning* of the complex number $c + di$. *Either the broken path $OA + AP$ or the direct*

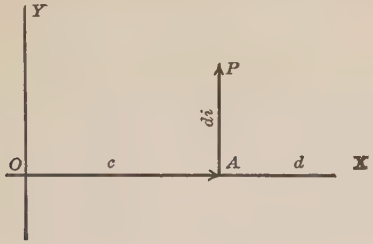


Fig. 137.—The geometrical construction of a complex number, $c + di$.

path OP may be taken as the representation of $c + di$, and either path constitutes **the definition of the sum of c and di.**

In the same manner $c - di$, $-c - di$, and $-c + di$ may be constructed.

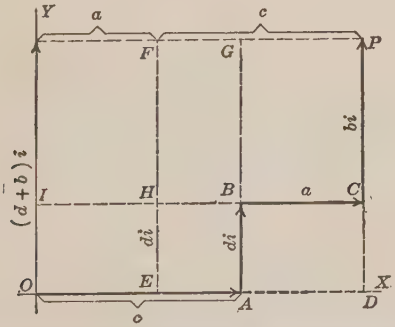


Fig. 138.—Illustration of the application of the laws of algebra and the expression $c + di + a + bi$.

The meaning of some of the laws of algebra as applied to imaginaries may now be illustrated. Let us construct

$$c + di + a + bi.$$

The first two terms, $c + di$, give $OA + AB$, locating B (Fig. 138). The next two terms, $a + bi$, give $BC + CP$, locating P. Hence the entire expression locates the point P with reference to

O. Now if the original expression be changed in any manner allowed by the laws of algebra, the result is merely a different path to the same point. Thus:

$c + a + di + bi$ is the path OA, AD, DC, CP.
$(c + a) + (d + b)i$ is the path OD, DP
$a + di + c + bi$ is the path OE, EH, HC, CP.
$a + di + bi + c$ is the path OE, EH, HF, FP, etc.

The student should consider other cases. Is there any method of locating P with the same four elements, which the figure does not illustrate?

181. Laws.—It can be shown by simple geometrical construction that the operator i, as defined above, obeys the ordinary laws of algebra. We can then apply all of the elementary laws of algebra to the symbol i and work with it just as we do with any other letter. The following are illustrations of each law:

Commutative Law:

$$c + di + a + bi = c + a + di + bi = di + c + bi + a, \text{ etc.}$$
$$ai = ia, \qquad iai = iia = aii, \text{ etc.}$$

Thus the equation $10\sqrt{-1} = \sqrt{-1}\cdot 10$ or, better, $10\sqrt{-1}\cdot 1 = \sqrt{-1}\cdot 10\cdot 1$ may be said to mean that the result of performing the operation of turning unity through 90° and performing upon the result the operation of taking it 10 times, is the same as the result of performing the operation of taking unity 10 times and performing upon this result the operation of turning through 90°.

Associative Law:

$$(c + di) + (a + bi) = c + (di + a) + bi, \text{ etc}$$
$$(ab)i = a(bi) = abi, \text{ etc.}$$

Distributive Law:

$$(a + b)i = ai + bi, \text{ etc.}$$

The expression $\sqrt{-a}$, where a is any number of the arithmetical scale, is defined as equivalent to $\sqrt{-1}\cdot\sqrt{a}$, that is, $\sqrt{-a} = i\sqrt{a}$. For example, $\sqrt{-4} = 2i$, $\sqrt{-3} = i\sqrt{3}$, etc. *In what follows it is presupposed that the student will reduce expressions of the form* $\sqrt{-a}$ *to the form* $i\sqrt{a}$ *before performing algebraic operations.* From this it follows that $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$ *and not* $\sqrt{ab}$.

The relation $\sqrt{-4} = 2\sqrt{-1}$ may be interpreted as follows: -4 is the operator that quadruples and reverses; then $\sqrt{-4}$ is an operator which used twice quadruples and reverses. But $2\sqrt{-1}$ is an operator such that two such operators quadruple and reverse. That is, $\sqrt{-4} = 2\sqrt{-1}$.

182. Powers of i.—We shall now interpret the powers of i by means of the new significance of an exponent and by the commutative, associative, and other laws. First:

$$\begin{aligned} i^0 \text{ or } i^0\,1 &= +1 & i^5 &= i^4 i = i \\ i^1 \text{ or } i^1\,1 &= i & i^6 &= i^5 i = -1 \\ i^2 &= -1 & i^7 &= i^6 i = -i \\ i^3 = i^2 i &= -i & i^8 &= i^7 i = +1 \\ i^4 = i^2 i^2 &= +1 & \text{etc.} & \qquad \text{etc.} \end{aligned}$$

Whence it is seen that all even powers of i are either $+1$ or -1, and all odd powers are either i or $-i$. The student may reconcile this with Fig. 136. The zero power of i must be unity, for the exponent zero can only mean that the operation denoted by the symbol of operation *is not to be performed at all;* that is, unity is to be left unchanged; thus 10^0 or $10^0 \cdot 1 = 1$, $2^0 = 1$, $\log^0 x = x$, $\sin^0 x = x$, etc.

Exercises

Select as unit a distance ½ in. in length extending to the right and represent graphically each of the following expressions:

1. $i + 2i^2 + 3i^3 + 4i^4 + \ldots$

2. $i + i^4 + i^5 + i^8 + i^9 + \ldots$

3. $i + i^4 + i^7 + i^8 + i^9 + i^{12} + \ldots$

4. $i(i + i^4 + i^7 + i^8 + i^9 + i^{12} + \ldots)$.

5. $i + i^2 + i^3 + 2i^4 + i^5 + i^6 + i^7 + 3i^8 + \ldots$

183. Conjugate Complex Numbers.—Two complex numbers are said to be **conjugate** if they differ only in the sign of the term containing $\sqrt{-1}$. Such are $x + iy$ and $x - iy$.

Conjugate imaginaries have a real sum and a real product, for

$$\begin{aligned} (x + yi) + (x - yi) &= x + yi + x - yi, \\ &= x + x + yi - yi = 2x. \end{aligned}$$

Likewise, applying the ordinary rules of algebra,

$$(x + yi)(x - yi) = x^2 - y^2 i^2 = x^2 + y^2.$$

It is well to note that the *product of two conjugate complex numbers is always positive and is the sum of two squares.*

This fact is very important and will be used frequently. Thus,

$$(3 - 4i)(3 + 4i) = 3^2 + 4^2 = 25;\ (1 + i)(1 - i) = 2;$$
$$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1;\ \text{etc.}$$

184. *The sum, product, or quotient of two complex numbers is, in general, a complex number of the typical form* $a + bi$.

Let the two complex numbers be $x + yi$ and $u + vi$.

1. Their sum is $(x + yi) + (u + vi)$
$$= (x + u) + (y + v)i$$

by the laws of algebra. This last expression is in the form $a + bi$.

2. Their product is $(x + yi)(u + vi)$
$$= x(u + vi) + yi(u + vi)$$
$$= xu + xvi + yui + yvi^2$$
$$= (xu - yv) + (xv + yu)i$$

by the laws of algebra. This last expression is in the form $a + bi$.

3. Their quotient is
$$\frac{x + yi}{u + vi} = \frac{(x + yi)(u - vi)}{(u + vi)(u - vi)}.$$

By the preceding, the numerator is of the form $a' + b'i$. By Sec. **183,** the denominator equals $u^2 + v^2$. Then the quotient equals
$$\frac{a' + b'i}{u^2 + v^2} = \frac{a'}{u^2 + v^2} + \frac{b'}{u^2 + v^2}i$$

by distributive law. This last expression is of the form $a + bi$.

Exercises

Reduce the following expressions to the typical form $a + bi$; the student must change every imaginary of the form $\sqrt{-a}$ to the form $i\sqrt{a}$.

1. $\sqrt{-25} + \sqrt{-49} + \sqrt{-121} - \sqrt{-64} - 6i$.
2. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.
3. $(x - [2 + 3i])(x - [2 - 3i])$.
4. $(-5 + 12\sqrt{-1})^2$.
5. $(3 - 4\sqrt{-1})^2$.
6. $(\sqrt{1 + i})(\sqrt{1 - i})$.
7. $(\sqrt{e} - \sqrt{-e})^2$.

8. $\dfrac{a}{\sqrt{-1}}$.

9. $\dfrac{2}{3+\sqrt{-2}}$.

10. $\dfrac{56}{1-\sqrt{-7}}$.

11. $\dfrac{1+i}{1-i^2}$.

12. $\dfrac{1}{(1-i)^6}$.

13. $\dfrac{1-i^3}{(1-i)^3}$.

14. $\dfrac{1-2\sqrt{-3}}{1+2\sqrt{-3}}$.

15. $\dfrac{(2+3\sqrt{-1})^2}{2+\sqrt{-1}}$.

16. $\dfrac{a+xi}{a-xi}-\dfrac{a-xi}{a+xi}$.

185. *If a complex number be equal to zero, the imaginary and real parts are separately equal to zero.*

Suppose

$$x + y\sqrt{-1} = 0,$$

x and y being real numbers.

Then

$$x = -y\sqrt{-1}.$$

Now it is absurd or impossible that a real number should equal an imaginary, except they each be zero, since the real and imaginary scales are at right angles to each other and intersect only at the point zero.

Therefore

$$x = 0 \text{ and } y = 0.$$

If two complex numbers are equal, then the real parts and the imaginary parts must be respectively equal,

for if

$$x + yi = u + vi,$$

then

$$(x - u) + (y - v)i = 0.$$

Whence, by the above theorem,

$$x - u = 0 \text{ and } y - v = 0,$$

that is,

$$x = u \text{ and } y = v.$$

186. Modulus.—Let the complex number $x + yi$ be constructed, as in Fig. 139, in which $OA = x$ and $AP = yi$. Draw the line OP, and let the angle AOP be called θ.

The numerical length of OP is called the **modulus** of the complex number $x + yi$. It is algebraically represented by $\sqrt{x^2 + y^2}$, or by the symbol $|x + yi|$. Thus, mod $(3 + 4i) = \sqrt{9 + 16} = 5$.

The student can easily see that *two conjugate complex numbers have the same modulus.*

If $y = 0$, the mod $(x + yi) = \sqrt{x^2} = |x|$, where the vertical lines indicate that merely the numerical, or absolute, value of x is called for. Thus the modulus of any real number is the same as what is called the *numerical,* or *absolute* value, of the number. Thus mod $(-5) = 5$.

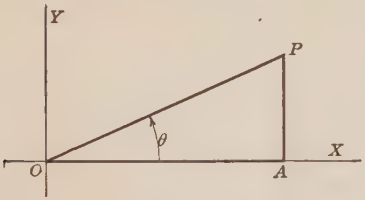


FIG. 139.—Modulus and amplitude of a complex number.

187. Amplitude.—In Fig. 139 the angle AOP or θ is called the **argument,** or **amplitude,** or simply the **angle,** of the complex number $x + yi$. Putting $r = \sqrt{x^2 + y^2} =$ mod $(x + yi) = |x + iy|$, we have

$$\sin\theta = \frac{y}{r} \text{ and } \cos\theta = \frac{x}{r}.$$

Therefore,

$$x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta). \qquad (1)$$

We have expressed the complex number $x + yi$ in terms of its modulus and amplitude. The last member of Eq. (1) is called the **polar form** of the complex number $(x + iy)$.

To put $3 - 4i$ in this form, we have

$$\text{mod } (3 - 4i) = \sqrt{9 + 16} = 5; \sin\theta = \frac{y}{r} = -\frac{4}{5}; \cos\theta = \frac{x}{r} = \frac{3}{5}.$$

Therefore,

$$(3 - 4i) = 5\left[\frac{3}{5} - \frac{4}{5}i\right].$$

The amplitude θ is $\tan^{-1}(-\frac{4}{3})$, and is in the *fourth* quadrant. Why?

It is well to plot the complex number in order to be sure of the amplitude θ. It avoids confusion to use positive angles in all cases. For example, to change $3 - \sqrt{3}i$ to the polar form, plot

the point $(3, -\sqrt{3})$ and find from the triangle that $r = 2\sqrt{3}$ and $\theta = 330°$. Hence

$$3 - \sqrt{3}i = 2\sqrt{3}\ (\cos 330° + i \sin 330°).$$

The amplitude of all real positive numbers is 0, and of all real negative numbers is 180°. Unity, expressed in terms of its modulus and amplitude, is, evidently, $1\ (\cos 0 + i \sin 0)$. It is, moreover, clear that this is not the only way in which unity may be expressed in polar form, since its amplitude may as well be taken equal to 360°, 720°, or, in general, any integral multiple of 360°.

188. Vector.—The point P (Fig. 139) located by $OA + AP$, or $x + yi$, may also be considered as located by the line or radius vector OP; that is, by a line starting at O, of length r, and making an angle θ with the direction OX. A directed line, as we are now considering OP, is called a **vector.** When thus considered, the two parts of the compound operator

$$r\ (\cos \theta + i \sin \theta) \qquad (1)$$

receive the following interpretation: The operator $(\cos \theta + i \sin \theta)$, which depends upon θ alone, turns the unit lying along OX through an angle θ, and may, therefore, be looked upon as a *versor* of rotative power θ. The versor $(\cos \theta + i \sin \theta)$ is often abbreviated by the convenient symbol **cis** θ. The operator r is a tensor, which stretches the turned unit in the ratio $1 : r$. The result of these two operations is that the point P is located r units from O in a direction making the angle θ with OX.

Thus, the operator $(\cos \theta + i \sin \theta)$ is simply a more general operator than i, but of the same kind. The operator i turns a unit through a right angle and the operator $(\cos \theta + i \sin \theta)$ turns a unit through an angle θ. If θ be put equal to 90°, $\cos \theta + i \sin \theta$ reduces to i.

For

$\theta = 0°$, $\cos \theta + i \sin \theta$ reduces to 1
$\theta = 90°$, $\cos \theta + i \sin \theta$ reduces to i
$\theta = 180°$, $\cos \theta + i \sin \theta$ reduces to -1
$\theta = 270°$, $\cos \theta + i \sin \theta$ reduces to $-i$.

Since $3 - 4i = 5(\frac{3}{5} - \frac{4}{5}i)$, the point located by $3 - 4i$ may be reached by turning the unit vector through an angle $\theta = \sin^{-1}(-\frac{4}{5}) = \cos^{-1}\frac{3}{5}$ and stretching the result in the ratio 1:5.

If a complex number vanishes, its modulus vanishes; and, conversely, if the modulus vanishes, the complex number vanishes.

If $x + yi = 0$, then $x = 0$ and $y = 0$, by Sec. **186.** Therefore, $\sqrt{x^2 + y^2} = 0$. Also, if $\sqrt{x^2 + y^2} = 0$, then $x^2 + y^2 = 0$, and since x and y are real, neither x^2 nor y^2 is negative, and so their sum is not zero unless each be zero.

If two complex numbers are equal, their moduli are equal, but if two moduli are equal, the complex numbers are not necessarily equal.

If $x + yi = u + vi$, then $x = u$ and $y = v$ by Sec. **186.**

Therefore,

$$\sqrt{x^2 + y^2} = \sqrt{u^2 + v^2}.$$

But if

$$\sqrt{x^2 + y^2} = \sqrt{u^2 + v^2},$$

obviously x^2 need not equal u^2 nor $y^2 = v^2$.

189. Sum of Complex Numbers.—Let a given complex number locate the point A (Fig. 140) and let a second complex number locate the point B. Then if the first of the complex numbers be represented by the radius vector OA, and if the second complex number be represented by the radius vector OB, the sum of the two complex numbers will be represented by the diagonal OC of the parallelogram constructed on the lines OA and OB. This law of addition is the well-known *law of addition of vectors* used in physics when the resultant of two forces or the resultant of two velocities, two accelerations, or two directed magnitudes of any kind is to be found.

The proof that the sum of the two complex numbers is represented by the diagonal OC is very simple. Let the graph of the first complex number be $OD_1 + D_1A$ and let that of the second be $OD_2 + D_2B$. To add these, at the point A construct $AE = OD_2$ and $EC = D_2B$. Then the sum of the two complex numbers is geometrically represented by $OD_1 + D_1A + AE + EC$, or by the radius vector OC which joins the end points. Since, by construction, the triangle AEC is equal to the triangle OD_2B, AC must be equal and parallel to OB, and the figure $OACB$ is a parallelogram.

OC, which represents the required sum, is the diagonal of this parallelogram, which we were required to prove.

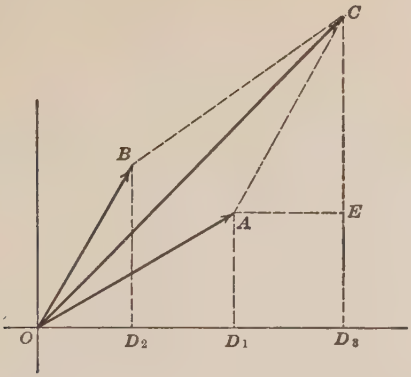


FIG. 140.—Sum of two complex numbers.

Exercise

Find algebraically the sum of the following complex numbers, and construct the same by means of the law of addition of vectors:

1. $(1 + 2i) + (3 + 4i)$.
2. $(1 + i) + (2 + i)$.
3. $(1 - i) + (1 + 2i)$.
4. $(3 - 4i) - (3 + 4i)$.
5. $(-2 + i) + (0 - 4i)$.
6. $(-1 + i) + (3 + i) + (2 + 2i)$.
7. $(2 - i) + (-2 + i) + (1 + i)$.

8. Find the modulus and the amplitude (in degrees and minutes) of $2(\cos 30° + i \sin 30°) + (\cos 45° + i \sin 45°)$.

9. By the parallelogram of vectors, show that the sum of two conjugate complex numbers is real.

10. If R be the sum of the complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$, etc., show that $-R$, z_1, z_2, z_3, . . . form the sides of a closed polygon.

190. Product of Complex Numbers.—*The product of two or more complex numbers is a complex number whose modulus is the product of the moduli and whose amplitude is the sum of the amplitudes of the complex numbers.*

Let the complex numbers be

$$z_1 = x_1 + y_1 i = r_1(\cos \theta_1 + i \sin \theta_1)$$
$$z_2 = x_2 + y_2 i = r_2(\cos \theta_2 + i \sin \theta_2), \text{ etc.}$$

By actual multiplication, we get $z_1z_2 =$

$$r_1r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)i]$$
$$= r_1r_2\,[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

Whence it is seen that r_1r_2 is the modulus of the product and $(\theta_1 + \theta_2)$ is the amplitude.

The above theorem is illustrated by Fig. 141. If the two given complex numbers be represented by their vectors OP_1 and OP_2, their product will be represented by the vector OP_3, whose direction angle

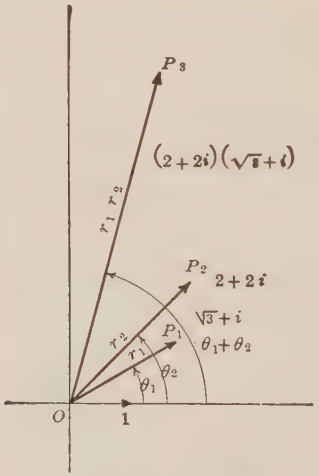


Fig. 141.—Product of two complex numbers.

is the sum of the amplitudes of the two given factors, and whose length OP_3 is the product of the lengths OP_1 and OP_2.

The figure represents the product $(2+2i)(\sqrt{3}+i)$. Expressed in terms of modulus and amplitude, these may be written

$$\sqrt{3} + i = 2\,(\cos 30° + i\sin 30°).$$
$$2 + 2i = 2\sqrt{2}\,(\cos 45° + i\sin 45°).$$

Hence,

$$r_1 = 2,\ r_2 = 2\sqrt{2},\ \theta_1 = 30°,\ \theta_2 = 45°.$$

Therefore,

$$(2 + 2i)(\sqrt{3} + i) = 4\sqrt{2}\,(\cos 75° + i\sin 75°).$$

Exercises

Find the moduli and amplitudes of the following products, and construct the factors and the products graphically. Take a positive angle for the amplitude in every case.

1. $(1 + \sqrt{3}i)(2\sqrt{3} + 2i)$.
2. $(2 + \frac{2}{3}\sqrt{3}i)(2 + 2i)$.
3. $(\sqrt{3} + 3i)(2 - 2i)$.
4. $(1 + i)^2$.
5. $(2 - 2\sqrt{3}i)(\sqrt{3} + 3i)$.
6. $(1 - i)^4$.
7. $(1 + i)^2(1 - i)^2$.

8. $2(\cos 15° + i \sin 15°) \times 3(\cos 25° + i \sin 25°)$. Find the numerical result by the use of the slide rule or trigonometric tables.

9. $2(\cos 10° + i \sin 10°) + (\frac{1}{3})(\cos 12° + i \sin 12°) \times 6(\cos 8° + i \sin 8°)$.

10. Find the value of $4\sqrt{2}(\cos 75° + i \sin 75°) \div (\sqrt{3} + i)$.

191. Quotient of Two Complex Numbers.—*The quotient of two complex numbers is a complex number whose modulus is the quotient of the moduli and whose amplitude is the difference of the amplitudes of the two complex numbers.* Let the complex numbers be

$$z_1 = x_1 + y_1 i = r_1(\cos \theta_1 + i \sin \theta_1).$$
$$z_2 = x_2 + y_2 i = r_2(\cos \theta_2 + i \sin \theta_2).$$

We have

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_2(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)}$$

$$= \frac{r_1[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]}{r_2(\cos^2 \theta_2 + \sin^2 \theta_2)}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

Whence it is seen that r_1/r_2 is the modulus of the quotient and $(\theta_1 - \theta_2)$ is the amplitude.

In Fig. 142, the complex number represented by the vector OP_1 when divided by the complex number represented by OP_2 yields the result represented by OP_3, whose length r_1/r_2 is found by dividing the length of OP_1 by the length of OP_2, and whose direction

angle is the difference $(\theta_1 - \theta_2)$ of the amplitudes of OP_1 and OP_2. The figure is drawn to scale for the case:

$$\frac{5\,(\cos 60° + i \sin 60°)}{2\,(\cos 20° + i \sin 20°)} = (2.5)\,(\cos 40° + i \sin 40°).$$

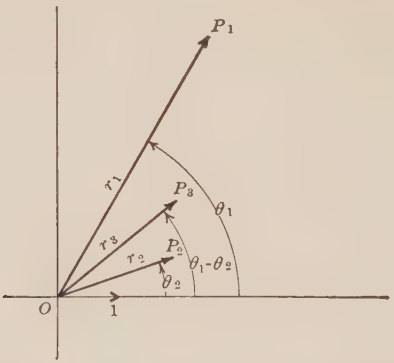


Fig. 142.—Quotient of two complex numbers.

Exercises

Find the quotient and graph the results in each of the following exercises. Always take amplitudes as positive angles and if $\theta_2 > \theta_1$, take $\theta_1 + 360°$ instead of θ_1.

1. $(1 + \sqrt{3}i) \div (\sqrt{2} + \sqrt{2}i)$. **3.** $(3\sqrt{3} - 3i) \div (-1 + \sqrt{3}i)$.
2. $(\frac{1}{2} + \frac{1}{2}\sqrt{3}i) \div (\sqrt{2} - \sqrt{2}i)$. **4.** $(1 - \sqrt{3}i) \div i$.
5. $2\,(\cos 36° + i \sin 36°) \div 5\,(\cos 4° + i \sin 4°)$.
6. $12\,(\cos 48° + i \sin 48°) \div [2\,(\cos 15° + i \sin 15°) \times 3\,(\cos 9° + i \sin 9°)]$.

7. $\dfrac{[4 + (\frac{4}{3})\sqrt{3}i](2 + 2\sqrt{3}i)}{8 + 8i}$.

8. Express in terms of a, b, c, d the amplitude of $(a + bi) \div (c + di)$.

192. De Moivre's Theorem.—As a special case of Sec. **190** consider the expression

$$(\cos \theta + i \sin \theta)^n,$$

where n is a positive number.

This being the product of n factors like $(\cos\theta + i\sin\theta)$, we write, by means of Sec. **190,**

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\ \ldots$$
$$= [\cos(\theta + \theta + \ldots) + i\sin(\theta + \theta + \ldots)],$$

or
$$(\mathbf{\cos\theta + i\sin\theta})^n = (\mathbf{\cos n\theta + i\sin n\theta}), \qquad (1)$$

which relation is known as De Moivre's theorem.

De Moivre's theorem holds for fractional values of n, for, first consider the expression

$$(\cos\theta + i\sin\theta)^{1/t},$$

where the power $1/t$ of $\cos\theta + i\sin\theta$ is, by definition, an operator such that the tth power of the expression equals $\cos\theta + i\sin\theta$.

Put $\theta = t\phi$, so that $\phi = \theta/t$.

Then
$$(\cos\theta + i\sin\theta)^{1/t} = (\cos t\phi + i\sin t\phi)^{1/t}$$
$$= [(\cos\phi + i\sin\phi)^t]^{1/t} \text{ by (1)}$$
$$= \cos\phi + i\sin\phi$$
$$= \cos\frac{\theta}{t} + i\sin\frac{\theta}{t}. \qquad (2)$$

Next consider the case in which $n = s/t$. We know

$$(\cos\theta + i\sin\theta)^{s/t} = [(\cos\theta + i\sin\theta)^s)]^{1/t}$$
$$= (\cos s\theta + i\sin s\theta)^{1/t} \text{ by (1)}$$
$$= \cos\frac{s\theta}{t} + i\sin\frac{s\theta}{t} \text{ by (2).} \qquad (3)$$

Likewise, the theorem may be proved for negative values of n

Illustration 1.—Find $(3 + i\sqrt{3})^4$.

Write $3 + i\sqrt{3} = 2\sqrt{3}(\cos 30° + i\sin 30°)$.

Then, by De Moivre's theorem,

$$(3 + i\sqrt{3})^4 = 144(\cos 120° + i\sin 120°)$$
$$= 144(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i)$$
$$= -72 + 72\sqrt{3}i.$$

Illustration 2.—Find $(2 + 2i)^{11}$.

Write $2 + 2i$ in the form

$$2 + 2i = 2\sqrt{2}(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i)$$
$$(2 + 2i)^{11} = (2\sqrt{2})^{11}(\cos 45° + i\sin 45°)^{11}$$
$$= (2\sqrt{2})^{11}(\cos 495° + i\sin 495°)$$
$$= (2\sqrt{2})^{11}(\cos 135° + i\sin 135°)$$
$$= (2\sqrt{2})^{11}(-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i)$$
$$= 2^{16}(-1 + i).$$

Exercises

Evaluate the following by De Moivre's theorem, using trigonometric table or slide rule when necessary.

1. $(8 + 8\sqrt{3}i)^{12}$.
2. $\sqrt[3]{27}\,(\cos 75° - i \sin 75°)$.
3. $\sqrt[3]{125i}$.
4. $[\cos 9° + i \sin 9°]^{10}$.
5. $(3 + \sqrt{3}i)^5$.
6. $[\frac{1}{2} + (\frac{1}{2})\sqrt{3}i]^4$.
7. $(1 + i)^8$.
8. $(-2 + 2i)^{\frac{1}{3}}$.
9. $[(\frac{1}{2})\sqrt{3} - (\frac{1}{2})i]^5$.
10. Find value of $(-1 + \sqrt{-3})^5 + (-1 - \sqrt{-3})^5$ by De Moivre's theorem.
11. Find the value of $x^2 - 2x + 2$ for $x = 1 + i$.
12. If $j_1 = -\frac{1}{2} + (\frac{1}{2})\sqrt{-3}$ and $j_2 = -\frac{1}{2} - (\frac{1}{2})\sqrt{-3}$, show that $j_1^3 = 1$, $j_2^3 = 1$, $j_1^2 = j_2$, $j_2^2 = j_1$, $j_1^{3n} = j_2^{3n} = 1$, $j_1^{3n+1} = j_1$.

193. The Roots of Unity.—In Sec. **187** the student has seen that unity may be written in any of the alternative forms:

$$1 = \cos 0 + i \sin 0$$
$$1 = \cos 2\pi + i \sin 2\pi$$
$$1 = \cos 4\pi + i \sin 4\pi$$
$$1 = \cos 6\pi + i \sin 6\pi$$
$$\cdots\cdots$$

and so on. By De Moivre's theorem the cube root of any of these is taken by dividing the amplitudes by 3. Therefore, from the above expressions in turn, there results

$$\sqrt[3]{1} = \cos 0 + i \sin 0 = 1.$$
$$\sqrt[3]{1} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = \cos 120° + i \sin 120°$$
$$= -\tfrac{1}{2} + i(\tfrac{1}{2})\sqrt{3}.$$
$$\sqrt[3]{1} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = \cos 240° + i \sin 240°$$
$$= -\tfrac{1}{2} - i(\tfrac{1}{2})\sqrt{3}.$$
$$\sqrt[3]{1} = \cos\left(\frac{6\pi}{3}\right) + i \sin \frac{6\pi}{3} = \text{same as first, etc.}$$

Therefore there are three cube roots of unity. Since these are the roots of the equation $x^3 - 1 = 0$, they might have been found by factoring, thus

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$
$$= (x - 1)(x + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i)(x + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}i).$$

The three roots of unity divide the angular space about the point O into three equal angles, as shown in Fig. 143. In the same way, it can be shown that there are four fourth roots, five fifth roots, etc., of unity and that the vectors representing them have modulus 1 and amplitudes that divide equally the space about O.

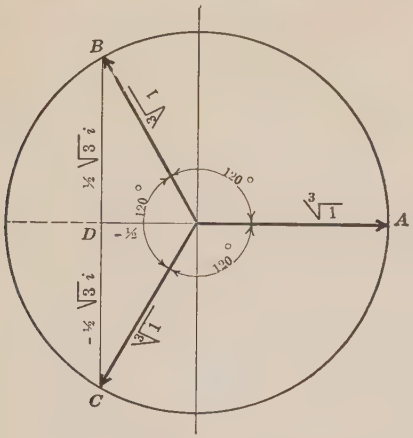


Fig. 143.—The cube roots of unity.

Illustration 1.—Find $\sqrt{\sqrt{3}+3i}$.

Write $\sqrt{3}+3i$ in the form

$$\sqrt{3}+3i = 2\sqrt{3}\,(\cos 60^\circ + i \sin 60^\circ).$$

Hence, by De Moivre's theorem,

$$\begin{aligned}(\sqrt{3}+3i)^{\frac{1}{2}} &= \sqrt[4]{12}\,(\cos 30^\circ + i \sin 30^\circ)\\ &= \sqrt[4]{12}\,[(\tfrac{1}{2})\sqrt{3} + (\tfrac{1}{2})i]\\ &= (\tfrac{1}{2})\sqrt[4]{108} + (\tfrac{1}{2})\sqrt[4]{12}i.\end{aligned}$$

A second root can be found by writing

$$\sqrt{3}+3i = 2\sqrt{3}\,[\cos (60^\circ + 360^\circ) + i \sin (60^\circ + 360^\circ)],$$

since adding a multiple of 360° to the amplitude does not change the value of the sine and cosine. In applying De Moivre's theorem, there results

$$\begin{aligned}(\sqrt{3}+3i)^{\frac{1}{2}} &= \sqrt[4]{12}\,(\cos 210^\circ + i \sin 210^\circ)\\ &= \sqrt[4]{12}\,[-(\tfrac{1}{2})\sqrt{3} - (\tfrac{1}{2})i].\end{aligned}$$

Illustration 2.—Find the cube root of $-\sqrt{2} + \sqrt{2}i$.

We write:

$$-\sqrt{2} + i\sqrt{2} = 2(\cos 135° + i \sin 135°)$$
$$= 2[\cos(135° + n360°) + i \sin(135° + n360°)].$$

in which n is any integer. Hence

$$(-\sqrt{2} + i\sqrt{2})^{1/3} = \sqrt[3]{2}[\cos(45° + n120°) + i \sin(45° + n120°)]$$
$$= \sqrt[3]{2}(\cos 45° + i \sin 45°) \text{ for } n = 0$$
$$= \sqrt[3]{2}(\cos 165° + i \sin 165°) \text{for } n = 1$$
$$= \sqrt[3]{2}(\cos 285° + i \sin 285°) \text{ for } n = 2.$$

These are the three cube roots of the given complex number. For $n = 3$ the first root is obtained a second time.

Exercises

Find all the indicated roots of the following:

1. $(8 + 8\sqrt{3}i)^{1/2}$.
2. $\sqrt[3]{27(\cos 75° + i \sin 75°)}$.
3. $\sqrt[3]{125i}$.
4. $(-2 + 2i)^{1/5}$.
5. $(2 + 2i)^{1/3}$.
6. $32^{1/5}$.
7. $\sqrt[9]{512}$.
8. Find to four places one of the imaginary seventh roots of $+1$.

NOTE: $\text{Cos } 51° 25.7' + i \sin 51° 25.7' = 0.6235 + 0.7818i$.

194*. Irrational Numbers.—A **rational number** is a number that can be expressed as the quotient of two integers. All other real numbers are **irrational.** Thus $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$, π, e, are irrational numbers. An irrational number is always intermediate in value to two rational numbers which differ from each other by a number as small as we please. Thus

$$1.414 < \sqrt{2} < 1.415$$
$$1.4142 < \sqrt{2} < 1.4143$$
$$1.41421 < \sqrt{2} < 1.41422, \text{ etc.}$$

It is easy to prove that $\sqrt{2}$ cannot be expressed as the quotient of two integers. For, if possible, let

$$\sqrt{2} = \frac{a}{b}, \tag{1}$$

where a and b are integers and a/b is in its lowest terms. Squaring the members of Eq. (1) we have

$$2 = \frac{a^2}{b^2}. \tag{2}$$

This cannot be true, since 2 is an integer and a and b are prime to each other

An irrational number, when expressed in the decimal scale, is never a repeating decimal, for, if the irrational number could be expressed in that manner, the repeating decimal could be evaluated by Sec. **112** in the fractional form $\frac{a}{1-r}$, which, by definition of an irrational number, is impossible. On the contrary, *every rational number when expressed in the decimal scale is a repeating decimal.* Thus $\frac{1}{3} = 0.33 \ldots$ and $\frac{1}{4} = 0.25000 \ldots$

The proof that π and e are irrational numbers is not given in this book.[1]

The student should not get the idea that, because irrational numbers are usually approximated by decimal fractions, the irrational number itself is not exact. This can be illustrated by the graphical construction of $\sqrt{2}$. Locate the point P whose coordinates are (1, 1). Call the abscissa OD and the ordinate DP. Then $OP = \sqrt{2}$, $OD = 1$, and $DP = 1$. It is obvious that the hypotenuse OP must be considered just as *exact* or definite as the legs OD and DP. The notion that irrational numbers are inexact must be avoided.

The process of counting objects can be carried out by use of the primitive scale of numbers 0, 1, 2, 3, 4, . . . The other numbers made use of in mathematics, namely

1. Positive and negative numbers
2. Integral and fractional numbers
3. Rational and irrational numbers
4. Real and imaginary numbers

may be looked upon as classes of numbers that permit the operations *subtraction*, *division*, and *evolution* to be carried out under all circumstances. Thus, in the history of algebra it was found that, in order to carry out subtraction under all circumstances, negative numbers were required; to carry out division under all circumstances, fractions were required; to carry out evolution of arithmetical numbers under all circumstances, irrational numbers were required; finally, to carry out evolution of algebraic numbers under all circumstances, imaginaries were required. It will be

[1] See "Monographs on Modern Mathematics," edited by J. W. A. Young.

found that it will not be necessary to introduce any additional form of number into algebra; that is, the most general number required is a number of the form $a + bi$, where a and b are positive or negative, integral or fractional, rational or irrational. This is the most general number that satisfies the following conditions:

1. The possibility of performing the operations of algebra and the inverse operations under all circumstances.

2. The conservation or permanence of the fundamental laws of algebra, namely, the commutative, associative, distributive, and index laws.

Further extension of the number system beyond that of complex numbers leads to operators which do not obey the commutative law in multiplication; that is, in which the value of a product is dependent upon the order of the factors, and in which a product does not necessarily vanish when one factor is zero. Numbers of this kind the student may later study in the introduction to the study of electromagnetic theory under the head of "Vector Analysis" or in the subject of "Quaternions." Such numbers or operators do not belong to the domain of numbers we are now studying.

195*. Simple Periodic Variation Represented by a Complex Number.—Fluctuating magnitudes exist that follow the *law* of S.H.M., although, strictly speaking, such magnitudes can be said to be "simple harmonic motions" in a figurative sense only. For example, we may think of the fluctuations of the voltage or amperage in an alternating current as following such a law. Thus, if E represent the electromotive force or pressure of the alternating current, then the fluctuations are expressed by

$$E = E_0 \sin \omega t$$

or by

$$E = E_0 \sin 2\pi f t,$$

where f is the frequency of the fluctuation. Instead of S.H.M. such a variable is more accurately called a **sinusoidal varying magnitude,** although for brevity we shall often call it S.H.M. The graph in rectangular coordinates of such a periodic function is often called a "wave," although this term should, in exact language, be reserved for *a moving periodic curve*, such as $y = a \sin (hx - kt)$.

If the polar representation

$$\rho = a \sin \omega(t - t_1) \qquad (1)$$

of the sinusoidal varying magnitude be used, then the graph of Eq. (1) is a circle of diameter a inclined the angular amount ωt_1 to the left of the axis OY, as is seen at once by calling $\omega t = \theta$ and $\omega t_1 = \alpha$ in the equation of the circle $\rho = a \sin(\theta - \alpha)$. The circle can be drawn when the length and the direction of its diameter are known; that is, the circle is completely specified when a and the direction of a (told by α) are given. Therefore the simple harmonic motion is completely symbolized by a *vector* OA of length a drawn from the origin in the direction given by the angle ωt_1. The direction angle of the vector OA is $\alpha + \pi/2$, or $\omega t_1 + \pi/2$.

The circle on the vector OA is located or characterized equally well if the rectangular coordinates (c, d) of the end of the diameter of the circle be given. But the complex number $c + di$ is represented by a vector which coincides with the diameter a of this circle. Hence, we may represent the circle by the complex number $c + di$. Its modulus is $a = \sqrt{c^2 + d^2}$ and its amplitude is $\alpha + \pi/2$. Therefore, if in Eq. (1) we take $a = \sqrt{c^2 + d^2}$, $\omega t_1 = \alpha$, and the variable angle $\omega t = \theta$, we can completely describe the S.H.M. by the complex number $c + di$. In the theory of alternating currents the sinusoidal varying current or voltage can conveniently be represented by a complex number, and that method of representing such magnitudes is in common use.

One of the advantages of representing S.H.M. by a vector or by a complex number is the fact that two or more such motions of like periods may then be compounded by the law of addition of vectors. This method of finding the resultant of two sinusoidal varying magnitudes of like periods possesses remarkable utility and simplicity.

To summarize, we may say:

1. *A sinusoidal varying magnitude is represented graphically in polar coordinates by a vector, which by its length denotes the amplitude and by its direction angle with respect to OY denotes the epoch angle.*

2. *Sinusoidal varying magnitudes of like periods may be compounded or resolved graphically by the law of parallelogram of vectors.*

If two sinusoidal varying magnitudes of like periods are in quadrature (that is, if their epoch angles differ by 90°), their relation, neglecting their epochs, can be completely expressed by a single complex number. Thus, let two S.H.M.'s in quadrature,

$$E_o = 113 \sin \omega(t - t_1) \tag{2}$$

and

$$E_c = 40 \cos \omega(t - t_1), \tag{3}$$

be represented by the circles and by the vectors marked OE_o and OE_c (Fig. 144). Call the resultant of these E_i. Then

$$E_i = 113 \sin \omega(t - t_1) + 40 \cos \omega(t - t_1) \quad (4)$$
$$= \sqrt{40^2 + 113^2} \sin \omega(t - t_2)$$
$$= 120 \sin \omega(t - t_2), \quad (5)$$

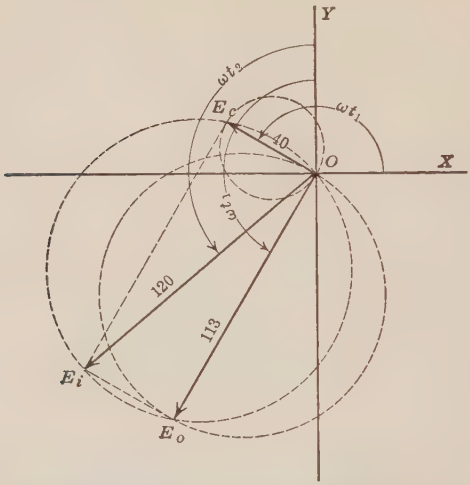


FIG. 144.—Composition of two S.H.M. in quadrature by law of addition of vectors.

where ωt_2 is measured as shown in Fig. 144. Instead of representing (2) and (3) in the polar diagram by OE_o and OE_c and their resultant by OE_i, we may represent (2), (3), and (4) in the complex number diagram (Fig. 145) by E_o, iE_c, and $E_o + iE_c$, respectively. Since the modulus and the amplitude of $E_o + iE_c$ are $\sqrt{E_o^2 + E_c^2}$ and α, respectively, and since the epoch angle of the resultant in Fig. 144 is $\omega t_2 = \omega t_1 - \alpha$, we can state the result as follows:

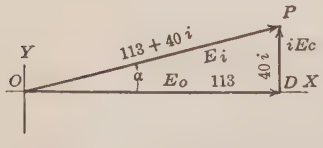


FIG. 145.—Complex number representation of the facts shown by polar diagram, Fig. 144.

If we have given two S.H.M.'s in quadrature and take the amplitude of the one possessing the greater epoch angle as c and the amplitude of the other S.H.M. as d, and construct the complex number $c + di$,

then this complex number $c + di$ *completely characterizes both of the S.H.M.'s and their resultant,* for we can determine the modulus ρ and the amplitude α of $c + di$ and then, if ωt_1 be the epoch angle of the motion with amplitude c, the epoch angle of the resultant is $\omega t_1 - \alpha$.

If we consider the two harmonic motions

$$\rho = a_1 \sin \omega(t - t_1)$$

and

$$\rho = a_2 \sin \omega(t - t_2),$$

and if t_1 be greater than t_2, the first S.H.M. reaches its maximum value *after* the second reaches its maximum. The first S.H.M. is therefore said to *lag* the amount $(t_1 - t_2)$ behind the second S.H.M. That is, a S.H.M. represented by a circle located anti-clockwise from a second circle represents a S.H.M. that *lags* behind the second.

196*. Illustration from Alternating Currents.—The steady current C flowing in a simple electric circuit is determined by the pressure or electromotive force E and the resistance R according to the equation known as Ohm's law,

$$C = \frac{E}{R},$$

or

$$E = CR.$$

E is the pressure or voltage required to make the current C flow against the resistance R. If the current, instead of being steady, varies or fluctuates, then the pressure CR required to make the current C flow over the true resistance is called the **ohmic voltage,** or ohmic pressure. But a changing or fluctuating current in an inductive circuit sets up a changing magnetic field around the circuit, from which there results a counter electromotive force, or choking effect, due to the changing of the current strength. This electromotive force is called the **reactive voltage** or **reactive pressure.** The choking effect that it has on the current is known as the **inductive reactance.** With a periodically changing current it acts alternately with and against the current. Opposite to the reactive voltage there is a component of the impressed voltage that is *consumed* by the reactance (see Fig. 146).

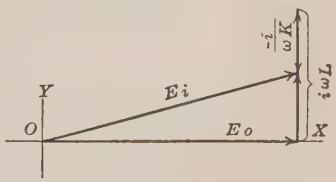


FIG. 146.—Complex number diagram of equation 5, Sec. **196.**

The pressure which is at every instant applied to the circuit from without is called the **impressed electromotive force,** or voltage.

Of the three pressures—namely, the impressed voltage, the ohmic voltage consumed by the resistance, and the reactive voltage consumed by the inductive reactance—any one may always be regarded as the resultant of the other two. Hence, if in a polar diagram the pressures be represented in magnitude and relative phase by the sides of a parallelogram, the impressed voltage may be regarded as the diagonal of a parallelogram of which the other two pressures are sides. Since, however, the reactance or the counter inductive pressure depends upon the rate of change of the current, it lags, in a sinusoidal current, 90° behind the true, or ohmic, voltage, which last is always in phase with the current. The pressure consumed by the counter inductive pressure therefore leads the current by 90°. Thus, in the language of complex numbers

$$E_i = E_o + iE_c, \tag{1}$$

in which

E_i = impressed pressure.
E_o = ohmic pressure, or pressure consumed by the resistance.
E_c = counter inductive pressure, or the pressure consumed by reactance.

It is found that the counter inductive pressure depends upon a constant of the circuit L called the **inductance** and upon the *angular velocity* or frequency of the alternating impressed pressure, so that

$$E_c = 2\pi fLC = \omega LC.$$

Hence Eq. (1) may be written

$$E_i = RC + i2\pi fLC \tag{2}$$
$$= RC + i\omega LC. \tag{3}$$

The modulus of the complex number on the right of this equation is

$$C\sqrt{R^2 + \omega^2 L^2}.$$

Considering, then, merely the absolute value $|E_0|$ and $|C|$ of pressure and current, we may write

$$|C| = \frac{|E_i|}{\sqrt{R^2 + \omega^2 L^2}}. \tag{4}$$

From the analogy of this to Ohm's law,

$$C = \frac{E}{R},$$

the denominator $\sqrt{R^2 + \omega^2 L^2}$ is thought of as limiting or restricting the current and is called the **impedance** of the circuit.

Let there be a condenser in the circuit of an alternator, but let the circuit be free from inductance. Then besides the pressure con-

sumed by the resistance, an additional pressure is required at any instant to hold the charge on the condenser. If K be the capacity of the condenser, it is found that that part of the pressure consumed in holding the charge on the condenser is $C/2\pi fK$, or $C/\omega K$, and is in phase position 90° behind the current C. The choking effect of this on the current may be called the **condensive reactance.** When a condenser is in the circuit *in addition to inductance*, the total pressure consumed by the reactance has the form:

$$2\pi fL - \frac{1}{2\pi fK},$$

and the complex number that symbolizes the vector is

$$E_i = RC + i2\pi fCL - \frac{iC}{2\pi fK} \tag{5}$$

(see Fig. 146).

Further illustrations of the applications of complex numbers to alternating currents is out of place in this book. The illustrations are merely for the purpose of emphasizing the usefulness of these numbers in applied science. An interesting application of the use of complex numbers to the problem of the steam turbine will be found in Steinmetz's "Engineering Mathematics," page 33.

Exercises

1. Draw the polar diagram and the complex-number representation of E_i if $R = 5$, $C = 21$, $f = 60$, $L = 0.009$, $K = 0.005$.

2. Draw a similar diagram if $E_i = 100$, $E_0 = 90$, $f = 40$, $L = 0.008$, $K = 0.003$.

CHAPTER XII

NORMAL EQUATION OF STRAIGHT LINE

197. Normal Equation of Straight Line.—The normal equation of the straight line was obtained in polar coordinates in Sec. **71.** The equation was written

$$\rho \cos (\theta - \alpha) = a. \tag{1}$$

In this equation ρ and θ are the polar coordinates of any point on the line, a is the distance of the line from the origin, and α is the direction angle of a perpendicular to the line from the origin (see Fig. 67). Expanding $\cos (\theta - \alpha)$ in Eq. (1) we obtain

$$\rho \cos \theta \cos \alpha + \rho \sin \theta \sin \alpha = a. \tag{2}$$

But for any value of ρ and θ, $\rho \cos \theta = x$ and $\rho \sin \theta = y$. Hence Eq. (2) may be written in rectangular coordinates

$$\mathbf{x \cos \alpha + y \sin \alpha = a.} \tag{3}$$

This also is called the **normal equation** of the straight line.

If an equation of any line be given in the form

$$ax + by = c, \tag{4}$$

it can readily be reduced to the normal form. Dividing this equation through by $\sqrt{a^2 + b^2}$,

$$\frac{\mathbf{a}}{\sqrt{\mathbf{a^2 + b^2}}}\mathbf{x} + \frac{\mathbf{b}}{\sqrt{\mathbf{a^2 + b^2}}}\mathbf{y} = \frac{\mathbf{c}}{\sqrt{\mathbf{a^2 + b^2}}}. \tag{5}$$

Now $a/\sqrt{a^2 + b^2}$ and $b/\sqrt{a^2 + b^2}$ may be regarded as the cosine and sine, respectively, of the angle formed with the positive X-axis by the line joining the origin to the point (a, b). Calling this angle α, Eq. (5) may be written

$$x \cos \alpha + y \sin \alpha = d, \tag{6}$$

which is of the form of Eq. (3) above. Inasmuch as the right-hand side of the equation in the normal form represents the distance of

the line from the origin, it is best to keep the right-hand side of the equation positive. The value of α and of the quadrant in which it lies is then determined by the signs of $\cos\alpha$ and $\sin\alpha$ on the left-hand side of the equation. The angle α may have any value from 0° to 360°.

Illustration 1.—Put the equation $3x - 4y = 10$ in the normal form. Here $a^2 + b^2 = 3^2 + (-4)^2 = 25$. Dividing by 5 we obtain

$$(\tfrac{3}{5})x - (\tfrac{4}{5})y = 2.$$

The distance of the line from the origin is 2. The angle α is the angle whose cosine is $\frac{3}{5}$ and whose sine is $-\frac{4}{5}$. Therefore, from the tables, $\alpha = 306° 52'$.

Illustration 2.—Put the equation $3x - 4y + 20 = 0$ in the normal form. Transposing and dividing by -1 to make the right-hand side of the equation positive, we obtain $-3x + 4y = 20$.

Here $\cos\alpha = -\frac{3}{5}$, $\sin\alpha = \frac{4}{5}$, $d = 4$. Hence $\alpha = 126° 52'$.

Illustration 3.—What is the distance between the lines (1) and (2)? The lines are parallel and on opposite sides of the origin. Their distance apart is therefore $2 + 4$ or 6.

Illustration 4.—Put $x + y = 1$ in the normal form. Here $\sqrt{a^2 + b^2} = \sqrt{2}$ The equation becomes $\frac{1}{2}\sqrt{2}x + \frac{1}{2}\sqrt{2}y = \frac{1}{2}\sqrt{2}$. $\alpha = 45°$, $d = \frac{1}{2}\sqrt{2}$.

Exercises

1. The shortest distance from the origin to a line is 5 and the direction angle of the perpendicular from the origin to the line is 30°. Write the equation of the line.

2. The perpendicular from the origin upon a straight line makes an angle of 135° with OX, and its length is $2\sqrt{2}$. Find the equation of the line.

3. Write the equation of a straight line in the normal form if $\alpha = 60°$ and $d = \sqrt{3}$.

4. Put $2\sqrt{3}x + 2y = 32$ in the normal form and find the numerical values of α and d.

5. Put $2x - 2y = 1$ in the normal form and find the values of α and d.

6. Find the equation of the straight line if the perpendicular from the origin on the line, makes an angle of 45° with the X-axis and its length is $\sqrt{2}$.

7. Put $x/4 + y/2 = 1$ in the normal form.

198. Additional Equations of the Straight Line.[1]—The equations of the straight line in the slope form

$$y = mx + b \tag{1}$$

and in the normal forms

$$\rho \cos (\theta - \alpha) = a \tag{2}$$

$$x \cos \alpha + y \sin \alpha = a \tag{3}$$

and the general form

$$ax + by + c = 0 \tag{4}$$

have already been used. Two constants and only two are necessary for each of these equations. The constants in the first equation are m and b; in the second and third, α and a; in the fourth, a/c and b/c, or any two of the ratios that result from dividing through by one of the coefficients. Equation (4) appears to contain three constants, but it is only the relative size of these that determines the particular line represented by the equation, since the line would remain the same when the equation is multiplied or divided through by any constant (not zero).

These facts are usually summarized by the statement that two conditions are necessary and sufficient to determine a straight line. The number of ways in which these conditions may be given is, of course, unlimited. Thus, a straight line is determined if we say, for example, that the line passes through the vertex of an angle and bisects that angle, or if we say that the line passes through the center of a circle and is parallel to another line, or if we say that the straight line is tangent to two given circles, etc. An important case is that in which the line is determined by the requirement that it pass through a given point in a given direction. The equation of the line adapted to this case is readily found. Let the given point be (x_1, y_1). The line through the origin with the required slope is

$$y = mx.$$

Translate this line so that it passes through (x_1, y_1) and we have

$$\mathbf{y - y_1 = m(x - x_1)}. \tag{5}$$

Another way of obtaining the same result is: Substitute the coordinates $(x_1\, y_1)$ in Eq. (1):

$$y_1 = mx_1 + b. \tag{6}$$

[1] See Sec. **19.**

Subtract the members of this from Eq. (1) above, so as to eliminate b. There results

$$y - y_1 = m(x - x_1). \tag{7}$$

This is the required equation. The given point is (x_1, y_1) and the direction of the line through that point is given by the slope m.

Another important case is that in which the straight line is determined by requiring it to pass through two given points. Let the second of the given points be (x_2, y_2). Substitute these coordinates in Eq. (5):

$$y_2 - y_1 = m(x_2 - x_1). \tag{8}$$

To eliminate m, divide the members of Eq. (7) by the members of Eq. (8)

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x}, \tag{9}$$

or, as it is usually written,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{10}$$

This is the equation of a line passing through two given points. Since Eq. (10) may be looked upon as a proportion, the equation may be written in a variety of forms.

Exercises

1. Find the equation of the line passing through (2, 3) with slope $\frac{2}{3}$.

2. Find the equation of the line passing through (2, 3), and (3, 5).

3. Find the line passing through (2, −1) making an angle whose tangent is 2 with the X-axis.

4. Find the line through (2, 3) parallel to $y = 7x + 11$.

5. A line passes through (−1, −3) and is perpendicular to $y - 2x = 3$. Find its equation.

6. Find the line passing through (−2, 3), and (−3, −1).

7. Find the equation of the line which passes through (−1, −3), and (−2, 4).

8. Find the slope of the line that passes through (−1, 6), and (−2, 8).

199. To Translate Any Locus a Given Distance in a Given Direction.—To move any locus the distance d to the right we sub-

stitute $x_1 - d$ for x in the equation of the locus. To move the locus the distance d in the y direction we substitute $y_1 - d$ for y. To move any locus the distance d in the direction α we substitute

$$\begin{aligned}(x_1 - d\cos\alpha) &\text{ for } x,\\ (y_1 - d\sin\alpha) &\text{ for } y,\end{aligned} \tag{1}$$

which must give the desired equation of the new locus. It is not necessary to use the subscript attached to the new coordinates if the distinction between the new and old coordinates can be kept in mind without this device.

The circle $x^2 + y^2 = a^2$ moved the distance d in the direction α becomes

$$(x - d\cos\alpha)^2 + (y - d\sin\alpha)^2 = a^2,$$

which may be changed to

$$x^2 - 2dx\cos\alpha + y^2 - 2dy\sin\alpha = a^2 - d^2.$$

200. Rotation of Any Locus.—It has already been explained that any locus can be rotated through an angle α by substituting $\theta_1 - \alpha$ for θ in the polar equation of the locus. It remains to determine the substitutions for x and y which will bring about the rotation of a locus in rectangular coordinates. Let us consider any point P of a locus before and after rotation through the given angle α. Call the coordinates of the point before rotation (x, y) in rectangular coordinates and (ρ, θ) in polar coordinates. Then, from Eqs. (1) and (2), Sec. **72,**

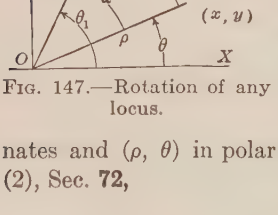


FIG. 147.—Rotation of any locus.

$$x = \rho\cos\theta \tag{1}$$

$$y = \rho\sin\theta. \tag{2}$$

Call the coordinates of the point after rotation (x_1, y_1) and (ρ_1, θ_1), but note that the value of ρ is unchanged by the rotation. Then for the point P' (Fig. 147), we may write

$$x_1 = \rho\cos\theta_1 \tag{3}$$

$$y_1 = \rho\sin\theta_1. \tag{4}$$

Since, however, the rotation requires that

$$\theta = \theta_1 - \alpha, \tag{5}$$

Eqs. (1) and (2) become

$$x = \rho \cos(\theta_1 - \alpha) = \rho \cos\theta_1 \cos\alpha + \rho \sin\theta_1 \sin\alpha. \quad (6)$$
$$y = \rho \sin(\theta_1 - \alpha) = \rho \sin\theta_1 \cos\alpha - \rho \cos\theta_1 \sin\alpha. \quad (7)$$

But, from Eqs. (3) and (4), $\rho \cos\theta_1$ and $\rho \sin\theta_1$ are the new values of x and y; hence, substituting in Eqs. (6) and (7) from Eqs. (3) and (4) we obtain

$$\mathbf{x} = \mathbf{x}_1 \cos\alpha + \mathbf{y}_1 \sin\alpha. \quad (8)$$
$$\mathbf{y} = \mathbf{y}_1 \cos\alpha - \mathbf{x}_1 \sin\alpha. \quad (9)$$

Hence, if the equation of any locus be given in rectangular coordinates, it is rotated through the positive angle α by the substitutions

$$\mathbf{x}\cos\alpha + \mathbf{y}\sin\alpha \text{ for } \mathbf{x}$$
$$\mathbf{y}\cos\alpha - \mathbf{x}\sin\alpha \text{ for } \mathbf{y}, \quad (10)$$

in which it is permissible to drop the subscripts if the context shows in each case whether we are dealing with the old x and y or with the new x and y.

If the required rotation is clockwise, or negative, we must replace α by $(-\alpha)$ in all of the above equations.

Whenever convenient, the equation of a curve should be taken in the polar form if it is required to rotate the locus.

Important Facts.—The following facts should be remembered by the student:

1. *To rotate a curve through* 90°, *change* x *to* y *and* y *to* $(-x)$.

2. *Rotation through any angle leaves the expression* $x^2 + y^2$ (*or any function of it*) *unchanged.* This is obvious since the circle $x^2 + y^2 = a^2$ is not changed by rotation about (0, 0).

3. *Rotation through* $+45°$ *changes* $2xy$ to $y^2 - x^2$.
   Rotation through $-45°$ *changes* $2xy$ to $x^2 - y^2$.

4. *Rotation through* $+45°$ *changes* $x^2 - y^2$ to $2xy$.
   Rotation through $-45°$ *changes* $x^2 - y^2$ to $-2xy$.

Statements 3 and 4 follow at once from consideration of the equations

$$x^2 - y^2 = a^2 \quad (1)$$
$$2xy = a^2 \quad (2)$$
$$y^2 - x^2 = a^2 \quad (3)$$
$$-2xy = a^2 \quad (4)$$

of the four hyperbolas bearing corresponding numbers 1, 2, 3, and 4 in Fig. 148. The proper substitution in any case can be remembered by thinking of the four hyperbolas of this figure.

5. *The degree of an equation of a locus cannot be changed by a rotation.* This follows at once from the fact that the equations of transformation (Eqs. (8) and (9)) are linear.

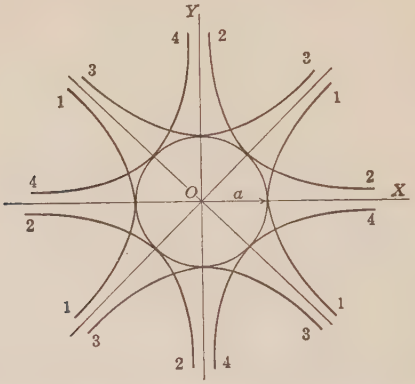


FIG. 148.—Effect of rotation on the special forms $x^2 + y^2$, $2xy$, and $x^2 - y^2$.

Exercises

In order to shorten the work, use statements 1 to 4 whenever possible.

1. Turn the locus $x^2 - y^2 = 4$ through 45°.

2. Turn $x^2 + y^2 = a^2$ through 79°. Turn $4xy = 1$ through 45°.

3. Turn $x \cos \alpha + y \sin \alpha = a$ through an angle β. (Since this locus is well known in the polar form, transformation Eqs. (6) and (7) above need not be used.)

4. Rotate $x^2 - y^2 = 1$ through 90°.

5. Rotate $x^2 - y^2 = a^2$ through −45°.

6. Rotate $x^2 - y^2 = 1$ through 30°.

7. Rotate $x^2 - y^2 = 4$ through 60°.

8. Rotate $4xy = 1$ through 30°.

9. Rotate $x^2 + 2y^2 = 1$ through 45°.

10. Change the equation $(x - a)^2 + (y - b)^2 = r^2$ to the polar form.

11. Change $\rho \cos 2\theta = 2a$, one of a class of curves known as Cote's spirals, to the Cartesian form.

12. Show that $\rho^2 - 2\rho\rho_1 \cos(\theta - \theta_1) + \rho_1^2 = a^2$ is the polar equation of a circle with center at (ρ_1, θ_1) and of radius a.

13. Write the Cartesian equation of the locus $\rho^2 = 16 \sin 2\theta$.

14. Turn $\rho^2 = 8 \sin 2\theta$ through an angle of 45°.

15. Rotate $x^2 - 2y^2 = 1$ through 90°.

16. Rotate $(x^2 + y^2)^{3/2} + (x^2 - y^2)^{3/2} = 1$ through 45°.

17. Rotate $\log(x^2 + y^2) = \tan(x^2 - y^2)$ through 45°.

18. Rotate $x^2 - 6xy + y^2 = 1$ through 45°.

19. Rotate $x^{1/2} + y^{1/2} = a^{1/2}$ through 45°. Show that the result is the parabola $y = x^2/a\sqrt{2} + a/2\sqrt{2}$, and sketch the curves.

201. Distance of Any Point from Any Line.—Let the equation of the line l (Fig. 149) in the normal form be

$$x \cos \alpha + y \sin \alpha = a, \tag{1}$$

and let (x_1, y_1) be any point P in the plane (see Fig. 149). If the point (x_1, y_1) is on the opposite side of the line from the origin, the line can be moved so that it will pass through the point by translating it the proper distance in the direction α. Let the unknown amount of the required translation be represented by d. To translate the line the amount d in the direction α, we must substitute for x and y the values

$$\begin{aligned} x &= x' - d \cos \alpha \\ y &= y' - d \sin \alpha. \end{aligned} \tag{2}$$

We obtain

$$(x' - d \cos \alpha) \cos \alpha + (y' - d \sin \alpha) \sin \alpha = a. \tag{3}$$

The line represented by this equation passes through the point (x_1, y_1). Substituting these coordinates for x' and y' and solving for d, we have

$$\mathbf{d = x_1 \cos \alpha + y_1 \sin \alpha - a.} \tag{4}$$

This is the distance of (x_1, y_1) from the given line.

If the given point is on the same side of the line as the origin, as is the point $P(x_2, y_2)$ (Fig. 149), then the given line must be translated the distance d in the direction $(180° + \alpha)$, and the result is the same as Eq. (4) above, except all signs are changed. We are usually interested only in the numerical value of d, so that Eq.

(4) may be used for all cases. When the value of d comes out negative it merely means that the given point is on the same side of the line as the origin.

Equation (4) may be interpreted as follows:

To find the distance of any point from a given line, put the equation of the line in the normal form, transpose all terms to the left-hand member and substitute the coordinates of the given point for x

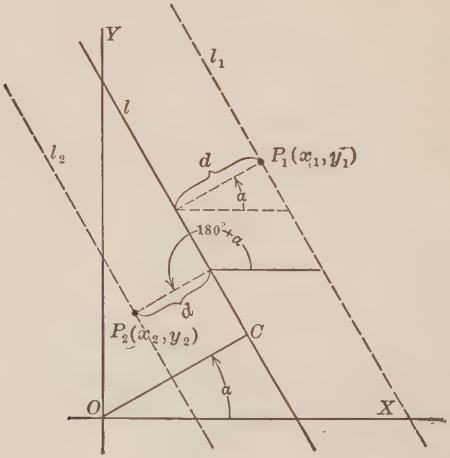


FIG. 149.—Distance of any point from a given line.

and y. The absolute value of the left-hand member is the distance of P from the line.

The above facts may be stated in an interesting form as follows: Let any line be

$$x \cos \alpha + y \sin \alpha - a' = 0.$$

If the coordinates of any point on this line be substituted in this equation, the left-hand member reduces to zero. If the coordinates of any point not on the line be substituted for x and y in the equation, the left-hand member of the equation does not reduce to zero, but becomes negative if the given point is on the origin side of the line and positive if the given point is on the non-origin side of the line. The absolute value of the left-hand mem-

ber in each case gives the distance of the given point from the line. Thus every line may be said to have a "positive side" and a "negative side." The "negative side" is the side toward the origin.

Illustration 1.—Find the distance of $(-1, 4)$ from the line $3x - 4y = 10$.

Transpose and put left-hand member in normal form

$$\tfrac{3}{5}x - \tfrac{4}{5}y - 2 = 0.$$

Substitute -1 for x and 4 for y. The left member is now the value of d, so that

$$d = -\tfrac{3}{5} - \tfrac{16}{5} - 2 = -5\tfrac{4}{5}.$$

The result is negative, so that $(-1, 4)$ is on the same side of the line as the origin.

Illustration 2.—Find the distance of $(2, -4)$ from

$$\frac{x-2}{4} = \frac{y+3}{7}.$$

Clear of fractions and simplify,

$$7x - 4y - 26 = 0.$$

Put in normal form,

$$\frac{7}{\sqrt{65}}x - \frac{4}{\sqrt{65}}y - \frac{26}{\sqrt{65}} = 0.$$

Substitute 2 for x and -4 for y,

$$d = \frac{4}{\sqrt{65}}.$$

The point is on the non-origin side of the given line, and $\frac{4}{\sqrt{65}}$ of one unit from it.

Exercises

1. Find the distance of the point (4, 5) from the line $3x + 4y = 10$.
2. Find the distance from the origin to the line $x/3 - y/4 = 1$.
3. Find the distance from $(-3, -4)$ to $12(x + 6) = 5(y - 2)$.
4. Find the distance from (3, 4) to the line $x/3 - y/4 = 1$.
5. Find the distance between the parallel lines $y = 2x + 3$, and $y = 2x + 5$.
6. Find the distance between $y = 2x - 3$, $y = 2x + 5$.
7. Find the distance from (0, 3) to $4x - 3y = 12$.
8. Find the distance from (0, 1) to $x + 2 - 2y = 0$.

CHAPTER XIII

LOCI

202. The Analysis of Problems.—The student has acquired, in the preceding chapters, a considerable amount of information concerning the elementary functions. He has likewise become acquainted with some properties of the graphs of these functions. Of more importance to him, however, is the ability to utilize this information by applying it to practical problems. It is not sufficient to gain a merely formal mastery of mathematics. Useful as it is to be able to find the roots of a given equation, or carry out an indicated computation, of much greater value is the power to analyze and solve new problems not stated in final mathematical form. The development of this analytical power is of special importance to one who requires his mathematics, as does the engineer, the physicist, or the chemist, for actual everyday use. Nature presents situations to be analyzed, not equations to be solved.

One mistake which the beginner often makes is the failure to understand the *whole* of a problem when he is capable of understanding *each part* of it. Such a student reads the problem through, is overcome by its length and apparent complexity, and reports that he "does not understand it," and probably that he "does not know how to start." In solving such a problem, it is often necessary to read it through several times; but this alone may not be sufficient. When he is ready to attempt the actual mathematical analysis of the problem, the student should read it from the beginning phrase by phrase, writing down in mathematical symbols the content of each phrase, as far as this may be possible. Consider, for example, the following simple problems:

Illustration 1.—A ship A is sailing south at the rate of 5 miles per hour: 2 hr. after it passes a certain point P, a ship B sails due east

through P at the rate of 8 miles per hour. Measuring time from the instant B passes through P, express the distance between the two ships as a function of t.

After reading this problem through until the general idea is clear, one should proceed as follows, translating the phrases into mathematical symbolism. "A ship A is sailing south"—one draws a vertical line and attaches a downward pointing arrowhead, marked A, to some point on this line. "At the rate of 5 miles per hour"—one puts down, near A, the figure, 5 miles/hr. "Two hours after it passes a certain point P"—one marks a point P on the line, above A. "A ship sails due east through P"—one draws a horizontal line through P and attaches a properly directed arrowhead, marked B, at some point on this line. "At 8 miles per hour"—one puts down, near B, the figure 8 miles/hr. "Measuring time from the instant B passes through P"—this phrase tells us that, at $t = 0$, A was a 2-hr. sail or 10 miles south of P; so that at any instant t, the distance A is south of P is given by the expression $10 + 5t$. It also says that, since B was at P at $t = 0$, the distance of B to the right of t at a time t is $8t$. Thus the vertical distance PA should be labeled $10 + 5t$, and the horizontal distance PB, $8t$. "Express the distance between the two ships as a function of t"—since the distance, say d, between A and B is clearly equal to the square root of the sum of the squares of the distances PA and PB, we have

$$d = \sqrt{\overline{PA}^2 + \overline{PB}^2} = \sqrt{(10 + 5t)^2 + 64t^2} = \sqrt{89t^2 + 100t + 100}.$$

Illustration 2.—Find the locus of a point equidistant from a fixed point.

Even so brief a problem may be decomposed into parts, simpler than the whole. When one says "the locus of a point" it is implied that the point in question is a variable point which would therefore be represented by the variable coordinates (x, y). To translate the fact that the point (x, y) is "equidistant from a fixed point" it is necessary to choose this fixed point. One might appropriately call the fixed point (a, b) or (x_0, y_0). Adopting the latter, one recalls that the distance between two points (x, y) and (x_0, y_0) is given by the square root of the sum of the squares of $(x - x_0)$ and $(y - y_0)$. Hence the equation

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \text{constant}$$

expresses the condition which the point (x, y) must satisfy if it remain at a constant distance from the point (x_0, y_0), and is thus the equation of the desired locus. It is, in fact, the equation of a circle about the

given fixed point. Would it have been any less general to take the point (0, 0) as the fixed point?

Illustration 3.—Find the locus of a point equidistant from two two fixed points.

Here the word "equidistant" is used in a different sense. It clearly means that the variable point, in any of its allowed positions, is the same distance from the two fixed points. A gain in simplicity, in this problem as in many similar ones, results from a sensible choice of the location of the two fixed points. If one simply chooses two random points (x_1, y_1) and (x_2, y_2) the resulting equation is more complicated than necessary, and more difficult to interpret. There is no loss of generality, as the student should convince himself, in choosing the origin as one point and a point on, say, the X-axis as the other. Even this, however, is not the most sensible choice. It is very important in any problem to take complete advantage of whatever symmetry the problem possesses. If the two points are chosen on the X-axis (on the Y-axis would do just as well) and at equal distances, say a, from the origin, they will then be symmetrically disposed with respect to the coordinate axes. Then the distances of any point (x, y) to these two points $(a, 0)$ and $(-a, 0)$ are $\sqrt{(x-a)^2+y^2}$ and $\sqrt{(x+a)^2+y^2}$. The problem demands that these be equal:

$$\sqrt{(x-a)^2+y^2} = \sqrt{(x+a)^2+y^2}.$$

Simplification of this result produces the equation $x = 0$; namely, the equation of the Y-axis. The locus of a point equidistant to two fixed points is hence the perpendicular bisector of the line joining the two points.

These three illustrations give the student suggestions as to how to proceed in "setting up" a problem. This chapter contains a variety of problems, making use of parametric equations, Cartesian coordinates, or polar coordinates. The student should avoid a superficial understanding of these problems, and should study them with a view toward development of the sort of analytical power discussed above. In working the exercises he should make full use of the information he has already gained, and should always draw clear, reasonably accurate sketches. Above all he should never permit himself to fail because he does not start. At least he should make some sort of a diagram, some sort of a start, translating into the sketch or into an equation some phrase

or phrases; and he will soon learn that, attacked a bit at a time, the difficulties of the more complicated problems disappear.

203. Parametric Equations.—The equation of a plane curve is ordinarily given by an equation in two variables, as has been amply illustrated by numerous examples in the preceding chapters. It is obvious that a curve might also be given by the two equations containing three variables, for, if the third variable be eliminated from the two equations, a single equation in two variables results. When it is desirable to describe a locus by means of two equations in three variables the equations are known as **parametric equations,** as has already been explained in Sec. **79.** Two of the variables usually belong to one of the common coordinate systems and the third is an extra variable called the *parameter.* In applied science the variable *time* frequently occurs as a parameter.

The parametric equations of the circle have already been written. They are:

$$x = a \cos \phi, \qquad (1)$$
$$y = a \sin \phi,$$

where the parameter ϕ is the direction angle of the radius vector to the point (x, y). Likewise, the parametric equations of the ellipse have been written

$$x = a \cos \phi, \qquad (2)$$
$$y = b \sin \phi,$$

and those of the hyperbola have been written

$$x = a \sec \phi, \qquad (3)$$
$$y = b \tan \phi.$$

In harmonic motion, the ellipse was seen to be the resultant of the two S.H.M.'s in quadrature

$$x = a \cos \omega t, \qquad (4)$$
$$y = b \sin \omega t.$$

Here the parameter t is *time.* The following problems and exercises contain further examples of the use of parametric equations.

Problem 1.—A circle[1] rolls without slipping within a circle twice the diameter. Show that any point attached to the moving circle describes an ellipse.

[1] "Circle" is here used in the sense of a "disk" or circular area and not in the sense of a "circumference."

Draw the smaller rolling circle in any position within the larger circle, and call the point of tangency T, as in Fig. 150. Since the smaller circle is half the size of the larger circle, the smaller circle always passes through O, and the line AB joining the points of intersection of the small circle with the coordinate axes is, for all positions, a diameter, since the angle AOB is a right angle.

If we can prove that the arc AT = the arc HT for all positions of T, then we shall have shown that, as the small circle rolls from an initial position with point of contact at H, the end A of the

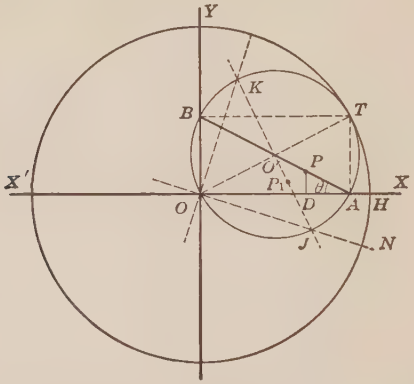


Fig. 150.—Generation of so-called "elliptic motion."

diameter AB slides on the line OX. Since B lies on OY and since AB is of fixed length, this will then prove, by Exercise 23, Sec. **79,** that any point of the small circle lying on the particular diameter AB describes an ellipse.

To prove that arc AT = arc HT, we have that the angle HOT is measured in radians by arc HT/OH. The angle $AO'T$ is measured in radians by arc $AT/O'A$. Since $\angle AO'T = 2\angle HOT$, we have

$$\frac{\text{arc } AT}{O'A} = 2\frac{\text{arc } HT}{OH}.$$

But, $OH = 2O'A$. Hence arc AT = arc HT.

We can now prove that any other point of the rolling circle describes an ellipse. Let any other point be P_1. Through P_1

draw the diameter $JO'K$. The above reasoning applies directly, replacing A by J and H by N.

It is easy to see that all points equidistant from the center of the small circle, such as the points P and P_1, describe ellipses of the same semi-axes a and b, but with their major axes variously inclined to OH.

Problem 2.—Determine the curve given by the parametric equations

$$x = a \cos 2\omega t \tag{1}$$
$$y = a \sin \omega t. \tag{2}$$

To eliminate t, the first equation may be written

$$x = a\,(1 - 2 \sin^2 \omega t). \tag{3}$$

From the second equation, $\sin \omega t = y/a$. Substituting for $\sin \omega t$ in Eq. (3),

$$x = a\left(1 - \frac{2y^2}{a^2}\right), \tag{4}$$

or

$$y^2 = -\frac{a}{2}x + \frac{a^2}{2}. \tag{5}$$

This curve is the parabola $y^2 = mx$, the special location of which the student should describe.

Problem 3. *The Cycloid.*—The **cycloid** is the curve traced by a point on the circumference of a circle, called the **generating circle,** which rolls without slipping on a fixed line called the **base.** To find the equation of the cycloid, let OA (Fig. 151) be the base, P the tracing point of the generating circle in any one position, and ϕ the angle between the radii SP and SH. Since P was at O when the circle began to roll,

$$OH = a\phi,$$

if a be the radius of the generating circle. Since $x = OD$ and $y = DP$, we have

$$\mathbf{x} = OH - SP \sin \phi = \mathbf{a}(\phi - \mathbf{sin}\ \phi). \tag{1}$$
$$\mathbf{y} = HS - SP \cos \phi = \mathbf{a}(\mathbf{1} - \mathbf{cos}\ \phi). \tag{2}$$

These are the parametric equations of the curve. For most purposes these are more useful than the Cartesian equation.

It is readily seen from the definition of the curve that the locus consists of an unlimited number of loops above the X-axis, with points of contact with the X-axis at intervals at $2\pi a$ (the circumference of the generating circle) and with maximum points at $x = \pi a$, $3\pi a$, etc.

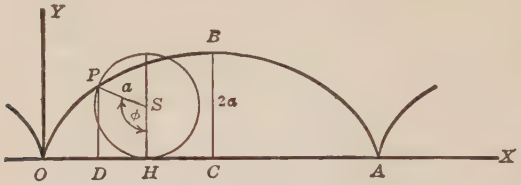


FIG. 151.—Definition of the cycloid.

Graphical Construction of the Cycloid.—To construct the cycloid (Fig. 152) draw a circle of radius 1.15 in. and divide the circumference into thirty-six equal parts. Draw horizontal lines through each point of division exactly as in the construction of the sinusoid (Fig. 57). Lay off uniform intervals of $\frac{1}{5}$ in. each on the X-axis, marked 1, 2, 3, . . . Then from the point of division of the circle p_1 lay off

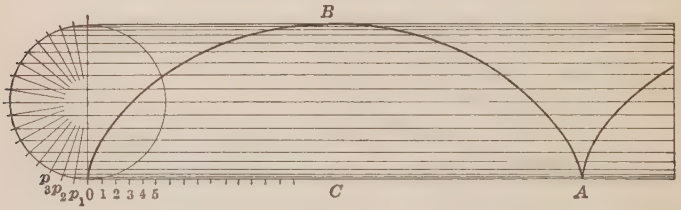


FIG. 152.—Construction of the cycloid.

the distance $O1$ to the right. From p_2 lay off $O2$ to the right, from p_3 lay off $O3$ to the right, etc. The points thus determined lie on the cycloid. The number of divisions of the circumference is, of course, immaterial except that an even number of divisions is convenient. Further, the divisions laid off on the base OA must be the same length as the arcs laid off on the circle.

Note that, by the process of construction above, the vertical distances from OX to points on the curve are proportional to $1 - \cos \phi$ and that the horizontal distances from OY to points on the curve are proportional to $\phi - \sin \phi$.

The analogy of the cycloid to the sine curve is brought out by Fig. 153. A set of horizontal lines are drawn as before and also a sequence of semi-circles spaced at horizontal intervals equal to the intervals of arc on the circle. The plane is thus divided into a large number of small quadrilaterals having two sides straight and two sides curved. Starting at O and sketching the diagonals of successive cornering quadrilaterals the cycloid is traced. If, instead of the sequence of circles, uniformly spaced vertical straight lines had been used, the sinusoid would have been drawn. The sinusoid on that account is frequently called the "*companion to the cycloid.*"

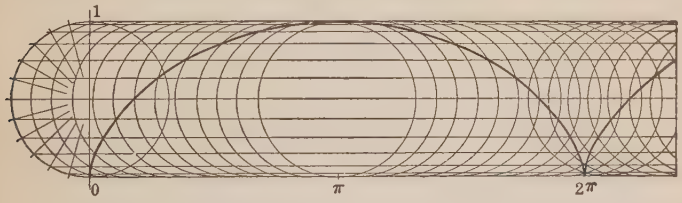


FIG. 153.—Analogy of the cycloid to the sinusoid.

Exercises

1. A point starts, at the instant $t = 0$, at the origin and moves so that its x-component of velocity is 3 in. per second, while its y-component of velocity is 2 in. per second. Write the parametric equations of the path of the point.

2. The x-component of velocity of a moving point is 3, while its y-component is -2. At the instant $t = 0$ the point is located at (3, 2). Write the parametric equations of its path.

3. The parametric equation of the path of a body thrown into the air with an initial velocity v_0 which makes an angle α with the horizontal are

$$x = (v_0 \cos \alpha)t,$$
$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t.$$

Eliminate the parameter t between these two equations, and show that the trajectory is a parabola with vertex at the point $[v_0^2 (\sin 2\alpha)/2g, v_0^2 (\sin^2 \alpha)/2g]$.

4. If a thread be wrapped around a circular disk of radius a, show that the parametric equations of the end point of the thread as it is unwrapped are (see Fig. 154)

$$x = a \cos \phi + a\phi \sin \phi.$$
$$y = a \sin \phi - a\phi \cos \phi.$$

5. Show that $x/a + y/b = 1$ is the Cartesian equation of the locus whose parametric equations are $x = a\cos^2\theta$, $y = b\sin^2\theta$. Figure 155 shows the method of constructing the loci, making use of the two circles $\rho = b\sin\theta$ and $\rho = a\cos\theta$. The student should satisfy himself as to the validity of the construction indicated in Fig. 155. Note that the Cartesian equation which formally results from the elimination of θ is not exactly equivalent to the parametric equations, since they permit only positive values of x and y. The parametric equations hence represent only that portion of $x/a + y/b = 1$ which lies in the first quadrant.

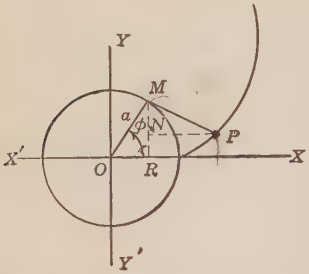


FIG. 154.—The curve traced by the end point of a thread as it is unwrapped off a circular disk.

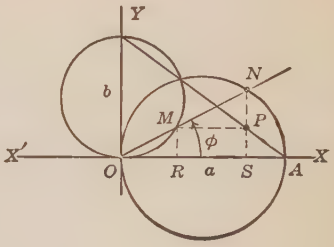


FIG. 155.—The construction of the locus $x = a\cos^2\theta$, $y = b\sin^2\theta$.

6. Show that the Cartesian equation of the locus $x = a\cos^3\phi$, $y = a\sin^3\phi$ is $x^{2/3} + y^{2/3} = a^{2/3}$.

7. Construct a cycloid by dividing a generating circle of radius 1.15 in. into twenty-four equal arcs and dividing the base into intervals $\frac{3}{10}$ in. each.

8. Compare the cycloid of length 2π and height 2 with a semi-ellipse of length 2π and height 2.

9. Write the parametric equations of a cycloid for origin C (Fig. 151).

10. Write the parametric equations of a cycloid for origin B (Fig. 151).

204. Trigonometric Loci. 1. *Graph of $y = \sin 2x$, $y = \sin nx$, etc.*—Since the substitution of nx for x in any equation multiplies the abscissas of the curve by $1/n$, or ($n > 1$) *shortens*, or contracts, the abscissas of all points of the curve in the uniform ratio $n:1$, the curve $y = \sin 2x$ must have twice as many crests, nodes, and

troughs in a given interval of x as the sinusoid $y = \sin x$. The curve $y = \sin 2x$ is, therefore, readily drawn from Fig. 57 as follows: Divide the axis OX into twice as many equal intervals as shown in Fig. 57 and draw vertical lines through the points of division. Then in the new diagram there are twice as many small rectangles as in the original. Starting at O and sketching the diagonals (curved to fit the alignment of the points) of successive cornering rectangles, the curve $y = \sin 2x$ is constructed. It is, of course, the orthographic projection of $y = \sin x$ upon a plane passing through the Y-axis and making an angle of 60° (the angle whose cosine is ½) with the xy-plane. The curve $y = \cos 2x$ is similarly constructed. In each of these cases we see that the period of the function is π and not 2π.

2. *Graph of $y = \sin^2 x$ and $y = \cos^2 x$.*—The graphs of $y = \sin^2 x$ and $y = \cos^2 x$ have important applications in science. For example, the heating effect of an alternating current $I = I_0 \sin \omega t$ depends upon the average value of I^2, that is to say, upon the average value of $\sin^2 \omega t$. By Eq. (5) of Chap. IX, $\cos^2 x = (1 + \cos 2x)/2$. Thus the curve $y = \cos^2 x$ is identical with the curve $2y - 1 = \cos 2x$, which is formed from the curve $y = \cos x$ by multiplying the abscissas by ½, translating up one unit, and multiplying the ordinates by ½. This curve can, therefore, be constructed as was the sinusoid in Chap. III, the stretching and the translation being accomplished by locating and numbering the axis as shown in Fig. 156. The fact should be noted that the square of a sine or cosine is sinusoidal in shape.

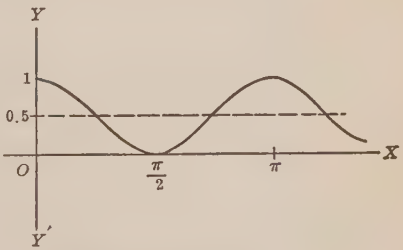


Fig. 156.—Graph of $y = \cos^2 x$.

3. *Graph of $\rho = \sin 2\theta$, $\rho = \cos 2\theta$, etc.*—The curve $\rho = \cos \theta$ is the circle of diameter unity coinciding in direction with the axis OX. We have already emphasized that, as θ varies from 0° to 360°, the *circle is twice drawn*, so that the curve consists of two superimposed circular loops. Now $\rho = \cos 2\theta$ (see Fig. 158) will

be found to consist of four loops, somewhat analogous to the leaves of a four-leafed clover, but each loop is described but once as θ varies from 0° to 360°. The curve $\rho = \cos 3\theta$ is a three-looped curve, but each loop is twice drawn as θ varies from 0° to 360°. Also, $\rho = \cos 11\theta$ has eleven loops, each twice drawn, while $\rho = \cos 12\theta$ has twenty-four loops, each one described but once, as θ varies from 0° to 360°.

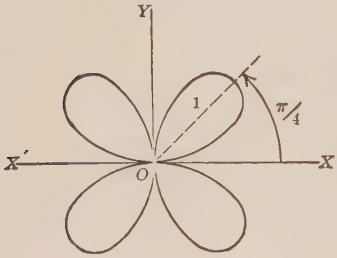


FIG. 157.—Graph of $\rho = \sin 2\theta$.

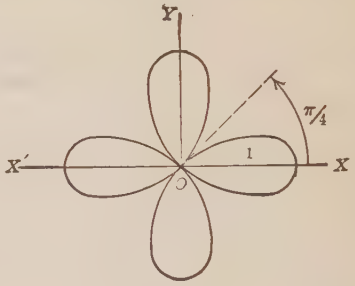


FIG. 158.—Graph of $\rho = \cos 2\theta$.

The curves $\rho = \cos 2\theta$, $\rho = \sin 3\theta$, $\rho = \sin \theta/2$ should be drawn by the student upon polar coordinate paper.

Exercises

1. How is the curve $y = \sin^2 x = \dfrac{1 - \cos 2x}{2}$ formed from the curve $y = \cos x$? Sketch the curve.

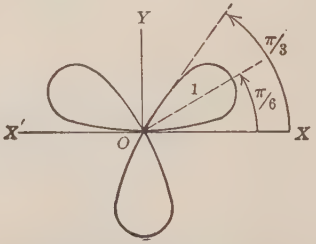


FIG. 159.—Graph of $\rho = \sin 3\theta$.

2. Justify the following statement: The curve $\rho = a \sin n\theta$ has $2n$ loops, all of which are distinct if n be even, but of which only n are distinct if n be odd, each of one of these n being twice drawn.

3. By how much does θ increase in describing one loop of the curve $\rho = a \cos 2\theta$, $\rho = a \sin 2\theta$, $\rho = a \sin 3\theta$, $\rho = a \sin 7\theta$, $\rho = a \sin n\theta$?

4. Draw the curves $\rho = \sin 4\theta$, $\rho = \cos 3\theta$.

5. Draw one loop of the curve $\rho = \sin 5\theta$.

6. Sketch the curves $\rho = \cos 4\theta$, $\rho = \sin 7\theta$.

7. What is the locus of the midpoint of the chords drawn through a given point on a circle?

8. All of the chords drawn through a given point on a circle are lengthened an amount a. What curve results?

205. Loci Defined by Focal Radii.—A number of important curves are defined by imposing conditions upon the distances of any point of the locus from two fixed points, called **foci.**

1. A point moves so that the product of its distances from two fixed points is constant. Find the equation of the path of the point

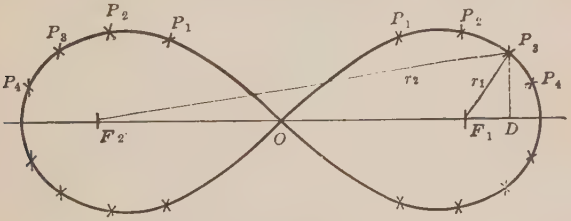


FIG. 160.—The lemniscate.

Let the two fixed points F_1 and F_2 (Fig. 160) be taken on the X-axis the distance a each side of the origin. Call the distances of P from the fixed points r_1 and r_2. Then the variables r_1 and r_2 in terms of x and y are

$$r_1^2 = y^2 + (x - a)^2.$$
$$r_2^2 = y^2 + (x + a)^2. \tag{1}$$

Hence

$$r_1^2 r_2^2 = [y^2 + (x - a)^2][y^2 + (x + a)^2]. \tag{2}$$

Calling the constant value of $r_1 r_2 = c^2$, we have, as the Cartesian equation of the locus,

$$[y^2 + (x - a)^2][y^2 + (x + a)^2] = c^4, \tag{3}$$

which may be written

$$(y^2 + x^2 + a^2)^2 - 4a^2x^2 = c^4. \tag{4}$$
$$(x^2 + y^2)^2 + 2a^2x^2 + 2a^2y^2 + a^4 - 4a^2x^2 = c^4 \tag{5}$$
$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2) + c^4 - a^4. \tag{6}$$

If $c = a$, the curve is called the **lemniscate,** and the Cartesian equation reduces to

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2). \tag{7}$$

For other values of c the curves are known as the **Cassinian ovals.** The student will show that when $c < a$ the curve consists of two separate ovals surrounding the foci, and for $c > a$ there is but a single oval.

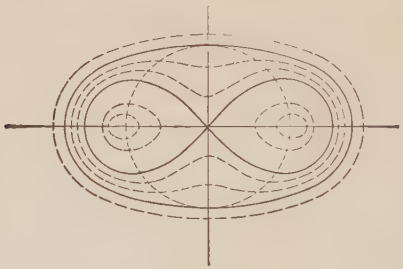


FIG. 161.—The lemniscate and the Cassinian ovals.

The curves are shown in Fig. 161. These curves give the form of the equipotential surfaces in a field around two positively or two negatively charged parallel wires.

2. Construct the curve such that the ratio of the distances of any point of the curve from two fixed points is constant. Let the two fixed

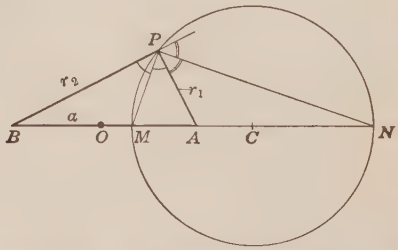


FIG. 162.—Construction of the curve $r_1/r_2 = m/n$, or the circle MPN.

points be A and B (Fig. 162); let the constant ratio of the distances of any point of the curve from the two fixed points be $r_1/r_2 = m/n$.

To find one point of the locus, draw circles from A and B as centers whose radii are in the ratio m/n. Let these circles intersect at the point P. At P bisect the angle between PA and PB internally and externally by the lines PM and PN respectively. The line AB is

then divided at M internally in the ratio $MA/MB = m/n$ and externally at N in the ratio $NA/NB = m/n$, because the bisectors of any angle of a triangle divide the base into segments proportional to the adjacent sides. Since the external and internal bisectors of any angle must be at right angles to each other, PM is perpendicular to PN for any position of P. Hence the locus of P is a circle, since it is the vertex of a right triangle described on the fixed hypotenuse MN.

If a large number of circles be drawn for different values of m/n, and if similar circles be described about B, then these circles are known as the **dipolar circles** (see Fig. 163). In physics it is found that these

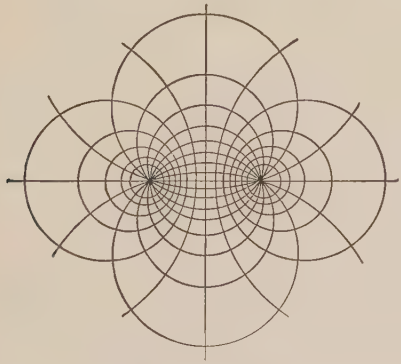


FIG. 163.—The dipolar circles, or a family of circles made by drawing $r_1/r_2 = e$ for various values of e.

circles are the equipotential lines about two parallel wires perpendicular to the plane of the paper at A and B and carrying electricity of opposite sign.

Exercises

1. Draw the locus satisfying the condition that the ratio of the distances of any point from two fixed points 10 units apart is $\frac{2}{3}$.

2. Draw the two circles which divide a line of length 14 internally and externally in the ratio $\frac{3}{4}$.

3. What is the locus of a point for which the difference of the squares of its distances from two fixed points is constant?

4. What is the locus of a point for which the sum of the squares of its distances from two fixed points is constant?

5. What is the locus of a point the sum of whose distances from two fixed points is constant?

General Exercises

1. A point moves so that its abscissa is always 4 times its ordinate. What is the equation of the path of the point?

2. On polar coordinate paper select the point (1, 1). (This means the point whose coordinates are 1 *cm.*, and 1 *radian.*) Starting at this point, a point moves so that the radius vector of the moving point is always equal to the vectorial angle. Sketch the curve. Write the polar equation of the curve.

3. A point moves so that one of its polar coordinates, the radius vector, varies directly as the other polar coordinate, the vectorial angle. Write the polar equation of such a curve. Does the curve go through the point (1, 1)?

4. A polar curve is generated by a point which starts at the point (1, 2) and moves so that the *increase* in the radius vector always equals the *increase* in the vectorial angle. Write the equation of the curve.

5. A polar curve is generated by a point which starts at the point (1, 2) and moves so that the increase in the radius vector varies directly as the increase in the vectorial angle. Write the equation of the curve.

6. A ball is thrown from a tower with a horizontal velocity of 10 ft. per second. It falls at the same time through a variable distance given by $s = 16.1\ t^2$, where t is the elapsed time in seconds and s is in feet. Find the equation of the curve traced by the ball.

7. The point P divides the line AB, of fixed length, *externally* in the ratio $a:b$, that is, so placed that $PA/PB = a/b$. If the line AB move with its end points always remaining on two fixed lines OX and OY at right angles to each other, then P describes an ellipse of semi-axes a and b.

8. If in the last problem the lines OX and OY are not at right angles to each other, the point P still describes an ellipse.

9. A point moves so as to keep the ratio of its distances from two fixed lines AC and BD constant. Prove that the locus consists of four straight lines.

10. A sinusoidal wave of amplitude 6 cm. has a node at +5 cm. and an adjacent crest at +8 cm. Write the equation of the curve.

11. The velocity of a simple wave is 10 m. per second. The period is 2 sec. Find the wave length and the frequency.

12. A polar curve passes through the point (1, 1) and the radius vector varies inversely as the vectorial angle. Plot the curve and write its equation. Consider especially the points where the vectorial

angle becomes infinite and where it is zero. Sketch the same function in rectangular coordinates.

13. Rectangles are inscribed in a circle of radius r. Express by means of an equation and plot: (a) the area, and (b) the perimeter of the rectangles as a function of the breadth.

14. Right triangles are constructed on a line of given length h as hypotenuse. Express and plot: (a) the area, and (b) the perimeter as a function of the length of one leg.

15. A conical tent is to be constructed of given volume V. Express and graph the amount of canvas required as a function of the radius of the base.

16. A closed cylindrical tin can is to be constructed of given volume V. Plot the amount of tin required as a function of the radius of the can.

17. A rectangular water tank lined with lead is to be constructed to hold 108 cu. ft. It has a square base and open top. Plot the amount of lead required as a function of the side of the base.

18. An open cylindrical water tank is to be made of given volume V. The cost of the sides per square foot is two-thirds the cost of the bottom per square foot. Plot the cost as a function of the diameter.

19. An open box is to be made from a sheet of pasteboard 12 in. square, by cutting equal squares from the four corners and bending up the sides. Plot the volume as a function of the side of one of the squares cut out.

20. The illumination of a plane surface by a luminous point varies directly as the cosine of the angle of incidence, and inversely as the square of the distance from the surface. Plot the illumination of a point on the floor 10 ft. from the wall, as a function of the height of a gas burner on the wall.

21. Using the vertical distances between corresponding points on the curves $y = \sin t$ and $y = -\sin t$ as ordinates and the vertical distances between corresponding points of $y = 2t$ and $y = t^2$ as abscissas, find the equation of the resulting curve.

22. What is the locus of the midpoint of the chords, drawn through the origin, of the parabola $y^2 = 4px$?

23. Lines are drawn from a fixed point to a given line. What is the locus of a point which divides these lines in the ratio $a:b$?

24. The base and the area of a triangle are constant. What is the locus of the vertex?

25. What is the locus of the center point of a rectangle inscribed in a given triangle?

26. Show that the following construction of the hyperbola $xy^2 = a^3$ is correct. On the $-X$-axis lay off $OC = a$. Connect C with any point A on the Y-axis. At C construct a perpendicular to AC cutting the Y-axis in B. At B erect a perpendicular to BC cutting the $+X$-axis at D. Through A draw a parallel to the X-axis and through D draw a parallel to the Y-axis. The two lines last drawn meet at P, a point on the desired curve.

27. Explain the following construction of the cubical parabola $a^2y = x^3$. Lay off OB on the $-Y$-axis equal to a. From B draw a line to any point C of the X-axis. At C erect a perpendicular to BC cutting the Y-axis at D. At D erect a perpendicular to CD cutting the X-axis at E. Lay off OE on the Y-axis. Then OE is the ordinate of a point of the curve for which the abscissa is OC.

28. Explain and prove the following construction of the semicubical parabola, $ay^2 = x^3$. Lay off on the $-X$-axis, $OA = a$. From A draw a parallel to the line $y = mx$, cutting the Y-axis in B. Erect at B a perpendicular to AB, cutting the X-axis at C, and at C erect a perpendicular to OC. The point of intersection with $y = mx$ is a point of the curve.

CHAPTER XIV

THE CONIC SECTIONS

206. The Focal Radii of the Ellipse.—Draw any ellipse with major and minor circles of radii a and b respectively, as in Fig. 164. Draw tangents, II' and KK', to the minor circle at the extremities of the minor axes and complete the rectangle $II'KK'$. The points F_1 and F_2, in which IK and $I'K'$ cut the major axis, are called the **foci** of the ellipse. From any point on the ellipse draw

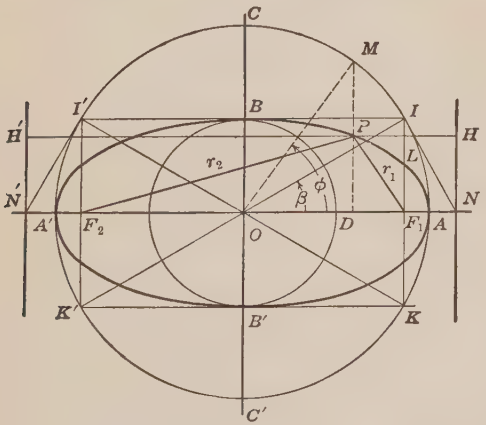


FIG. 164.

the focal radii $PF_1 = r_1$ and $PF_2 = r_2$, as shown in the figure. Represent the distance $OF_1 = OF_2$ by c. Then it follows from the triangle OIF_1 that

$$\mathbf{a^2 = b^2 + c^2.} \qquad (1)$$

This is one of the fundamental relations between the constants of the ellipse.

From the triangles PF_1D and PF_2D there follows:

$$r_1^2 = (c - x)^2 + y^2. \tag{2}$$

$$r_2^2 = (c + x)^2 + y^2. \tag{3}$$

But the equation of the ellipse is

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

or

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2). \tag{4}$$

Substituting this value of y^2 in Eq. (2)

$$r_1^2 = c^2 - 2cx + x^2 + \frac{b^2}{a^2}(a^2 - x^2) \tag{5}$$

$$= c^2 - 2cx + x^2 + b^2 - \frac{b^2}{a^2}x^2,$$

or, by Eq. (1),

$$r_1^2 = a^2 - 2cx + x^2\left[1 - \frac{b^2}{a^2}\right]. \tag{6}$$

Substituting

$$1 - \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2},$$

we obtain

$$r_1^2 = a^2 - 2cx + \frac{c^2x^2}{a^2} \tag{7}$$

$$= \left[a - \frac{cx}{a}\right]^2. \tag{8}$$

Therefore

$$r_1 = a - \frac{cx}{a}. \tag{9}$$

Likewise, from Eq. (3), by exactly the same substitutions, there follows

$$r_2 = a + \frac{cx}{a}. \tag{10}$$

From Eqs. (9) and (10), by addition,

$$\mathbf{r_1 + r_2 = 2a.} \tag{11}$$

Hence *in any ellipse the sum of the focal radii is constant and is equal to the major axis.*

The converse of this theorem, namely, *if the sum of the focal radii of any locus be constant, the curve is an ellipse*, can readily be proved. It is merely necessary to substitute the values of r_1 and r_2 from Eqs. (2) and (3) in Eq. (11), and simplify the resulting equation in x and y; or first square Eq. (11) and then substitute r_1 and r_2 from Eqs. (2) and (3). These results an equation of the *second degree lacking the term in* xy *and having the terms containing* x^2 *and* y^2 *both present and with coefficients of like signs*. By Sec. 81, such an equation represents an ellipse.

Hence the ellipse might have been defined as the *locus of a point, the sum of whose distances from two fixed points is constant.*

An ellipse can be drawn by attaching a string of length $2a$ by pins at the points F_1 and F_2 and tracing the curve by a pencil so guided that the string is always kept taut. Or, better, take a string of length $2a + 2c$ and form a loop enclosing the two pins; the entire curve can then be drawn with one sweep of the pencil.

The focal radii may also be evaluated in terms of the parametric or eccentric angle ϕ. The student may regard the following demonstration of the truth of Eq. (11) as simpler than that given above.

Since

$$x = a \cos \phi, \text{ and } y = b \sin \phi,$$

Eq. (2) gives

$$r_1^2 = b^2 \sin^2 \phi + (c - a \cos \phi)^2 \tag{12}$$

$$= b^2 \sin^2 \phi + c^2 - 2ac \cos \phi + a^2 \cos^2 \phi. \tag{13}$$

To put the right side in the form of a perfect square, write $b^2 = a^2 - c^2$. Then

$$r_1^2 = a^2 \sin^2 \phi - c^2 \sin^2 \phi + c^2 - 2ac \cos \phi + a^2 \cos^2 \phi$$

$$= a^2 - 2ac \cos \phi + c^2 \cos^2 \phi. \tag{14}$$

Whence

$$r_1 = a - c \cos \phi. \tag{15}$$

Likewise,

$$r_2 = a + c \cos \phi. \tag{16}$$

Whence

$$r_1 + r_2 = 2a.$$

207. The Eccentricity.—The ratio $c : a$ measures, in terms of a as unit, the distance of either focus from the center of the ellipse.

This ratio is called the **eccentricity** of the ellipse. In the triangle IF_1O, the ratio c/a is the cosine of the angle F_1OI, represented in what follows by β. Calling the eccentricity e, we have

$$\mathbf{e = \frac{c}{a} = \cos \beta.} \tag{1}$$

The ellipse is made from the major circle by contracting its ordinates in the ratio $m = b/a$, or by orthographic projection of the circle through the angle of projection

$$\alpha = \cos^{-1} \frac{b}{a}.$$

Hence, as companion to Eq. (1) we may write

$$m = \frac{b}{a} = \cos \alpha = \sin \beta. \tag{2}$$

208. The Ratio Definition of the Ellipse.—In Fig. 164, let the tangents to the major circle at I and I' be drawn. Draw a perpendicular to the major axis produced at the points cut by these tangents. These two lines are called the **directrices** of the ellipse.

We shall prove that the ratio PF_1/PH (or PF_2/PH') is constant for all positions of P. From Sec. **206,** Eq. (9) or (15),

$$r_1 = a - c \cos \phi. \tag{1}$$

From the figure,

$$ON = a \sec ION = a \sec \beta. \tag{2}$$

But

$$\cos \beta = \frac{c}{a}.$$

Hence,

$$ON = \frac{a^2}{c}. \tag{3}$$

But

$$PH = ON - x.$$

Therefore,

$$PH = \frac{a^2}{c} - a \cos \phi. \tag{4}$$

Hence, from Eqs. (1) and (4),

$$\frac{r_1}{PH} = \frac{PF_1}{PH} = \frac{a - c\cos\phi}{a^2/c - a\cos\phi} = \frac{c}{a}\left(\frac{a - c\cos\phi}{a - c\cos\phi}\right),$$

or

$$\frac{\mathbf{PF_1}}{\mathbf{PH}} = \frac{\mathbf{c}}{\mathbf{a}} = \mathbf{e} = \cos\beta. \tag{5}$$

A similar proof holds for the other focus and directrix. Thus for any point on the ellipse the distance to a focus bears a fixed ratio to the distance to the corresponding directrix. From Eq. (5), the ratio is seen to be less than unity.

Assuming the converse of the above, the ellipse might have been defined as follows: *The ellipse is the locus of a point whose distance from a fixed point (called the focus) is in a constant ratio, less than unity, to its distance from a fixed line (called the directrix).*

If, in any ellipse, $c = 0$, it follows that b must equal a and the ellipse reduces to a circle. If c is nearly equal to a, then, from the equation

$$a^2 = b^2 + c^2,$$

it follows that the semi-minor axis b must be very small. That is, for an eccentricity nearly unity the ellipse is very slender.

If the sun be regarded as fixed in space, then the orbits of the planets are ellipses, with the sun at one focus. (This is "*Kepler's First Law.*") The eccentricity of the earth's orbit is 0.017. The orbit of Mercury has an eccentricity of about 0.2, which is greater than that of any other planet.

Exercises

Find the eccentricities and the distance from center to foci of the following ellipses:

1. $x^2/9 + y^2/4 = 1$.
2. $y = \frac{2}{3}\sqrt{36 - x^2}$.
3. $25x^2 + 4y^2 = 100$.
4. $2y = \sqrt{1 - x^2}$.
5. $9x^2 + 16y^2 = 14$.
6. $2x^2 + 3y^2 = 1$.

Find the equation of the ellipse from the following data:

7. $e = \frac{1}{2}$, $a = 4$. Draw this ellipse.
8. $c = 4$, $a = 5$.
9. $r_1 = 6 - 2x/3$, $r_2 = 6 + 2x/3$.
10. $r_1 = 5 - 4\cos\phi$, $r_2 = 5 + 4\cos\phi$.

Solve the following exercises:

11. Find the eccentricity of the ellipse made by the orthographic projection of the circle $x^2 + y^2 = a^2$ through the angle 60°.

12. The angle of projection of a circle $x^2 + y^2 = a^2$ by which an ellipse is formed is α. Show that the eccentricity of the ellipse is $\sin \alpha$.

13. A circular cylinder of radius 5 is cut by a plane making an angle 30° with the axis. Find the eccentricity of the elliptic section.

14. If the greatest distance of the earth from the sun be 92,500,000 miles, find its least distance. (Eccentricity of earth's orbit = 0.017.)

15. In the ellipse $x^2/25 + y^2/16 = 1$, find the distance between the two directrices.

16. Write the equation of the ellipse whose foci are (2, 0), (−2, 0), and whose directrices are $x = 5$ and $x = -5$.

17. Prove Eq. 11, Sec. **206** by transposing one radical in:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a,$$

squaring, and reducing to an identity.

18. Obtain the equation of the ellipse from the definition near the end of Sec. **208.**

209. The Latus Rectum.—The double ordinate through the focus is called the **latus rectum** of the ellipse. The value of the semi-latus rectum is readily formed from the equation

$$y = \left(\frac{b}{a}\right)\sqrt{a^2 - x^2}$$

by substituting c for x. If l represents the corresponding value of y,

$$l = \left(\frac{b}{a}\right)\sqrt{a^2 - c^2} = \frac{b^2}{a}, \tag{1}$$

since $a^2 - c^2 = b^2$. Hence the entire latus rectum is represented by

$$2l = \frac{2b^2}{a}. \tag{2}$$

Equation (1) may also be written

$$l = b\sqrt{1 - \frac{c^2}{a^2}}$$

$$= b\sqrt{1 - e^2}. \tag{3}$$

In Fig. 164 the distances AF, AN, ON, OB, OF, FN may readily be expressed in terms of a and e as follows in Eqs. (4) to (10). The

addition of Eqs. (11), (12), and (13) brings into a single table all the important formulas of the ellipse.

$$\mathbf{AF_1} = a - c = \mathbf{a(1 - e)}. \tag{4}$$

$$\mathbf{AN} = \frac{AF_1}{e} = \frac{\mathbf{a(1 - e)}}{\mathbf{e}}. \tag{5}$$

$$\mathbf{ON} = a \sec \beta = \frac{\mathbf{a}}{\mathbf{e}}. \tag{6}$$

$$\mathbf{e} = \cos \beta. \tag{7}$$

$$\mathbf{OB} = \mathbf{b} = \mathbf{a} \sin \beta = \mathbf{a}\sqrt{\mathbf{1 - e^2}}. \tag{8}$$

$$\mathbf{OF_1} = \mathbf{c} = \mathbf{ae}. \tag{9}$$

$$\mathbf{F_1N} = ON - c = \frac{\mathbf{a(1 - e^2)}}{\mathbf{e}}. \tag{10}$$

$$\mathbf{l} = \frac{\mathbf{b^2}}{\mathbf{a}} = \mathbf{a(1 - e^2)} \tag{11}$$

$$\mathbf{r_1} = \mathbf{a - ex} = \mathbf{a - x} \cos \beta. \tag{12}$$

$$\mathbf{r_2} = \mathbf{a + ex} = \mathbf{a + x} \cos \beta. \tag{13}$$

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Exercises

1. Find the value in miles of OF_1 for the case of the earth's orbit.

2. Find the equation of an ellipse whose minor axis is 10 units and in which the distance between the foci is 10.

3. In the ellipse $y = (\frac{2}{3})\sqrt{36 - x^2}$ find the length of the latus rectum and the value of e.

4. The eccentricity of an ellipse is $\frac{3}{5}$ and the latus rectum is 9 units. Find the equation of the ellipse.

5. In (*a*) $x^2 + 4y^2 = 4$ and (*b*) $2x^2 + 3y^2 = 6$ find the latus rectum, the eccentricity, and the distances ON and AF.

6. Determine the eccentricities of the ellipses,

(*a*) $y^2 = 4x - (\frac{1}{2})x^2$ (*b*) $y^2 = 4x - 2x^2$.

7. Find the value of β for the earth's orbit. (Use the S functions of the logarithmic table.)

8. Find the equation of an ellipse whose latus rectum is 2 units and minor axis is 4.

9. The distance from the focus to the directrix is 16 units. The distance from the vertex to the nearest focus is 6 units. Find the equation of the ellipse.

10. The axes of an ellipse are known. Show how to locate the foci.

11 In an ellipse $a = 25$ ft., $e = 0.96$. What are the values of c and b?

12. For a certain comet (Tempel's) the semi-major axis of the elliptic orbit is 3.5, and $c = 1.4$ on a certain scale. For another comet (Enke's) $a = 2.2$, $e = 0.85$. Sketch the curves, taking 3 cm. or 1 in. as the unit of measure.

13. If $l = 7.2$ and $e = 0.6$, find c, a, b.

14. An ellipse, with center at the origin and major axis coinciding with the X-axis, passes through the points (8, 3) and (6, 4). Find the axes of the ellipse.

210. Focal Radii of the Hyperbola.—Construct a hyperbola from auxiliary circles of radii a and b, then the transverse axis of

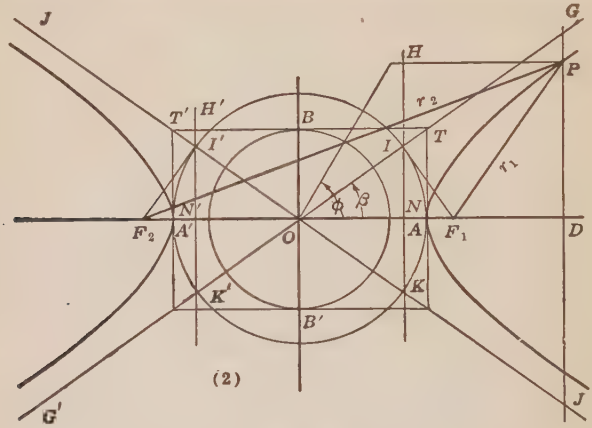

FIG. 165.

the hyperbola is $2a$ and the conjugate axis is $2b$. Unlike the case of the ellipse, b may be either greater or less than a. As previously explained, the asymptotes are the extensions of the diagonals of the rectangles $BTAO$, $BT'A'O$. From the points I, I', in which the asymptotes cut the a-circle, draw tangents to the a-circle. The points F_1, F_2 in which the tangents cut the axis of the hyperbola are called the **foci** (see Fig. 165).

The distance OF_1 or OF_2 is represented by the letter c. Then, since the triangles F_1IO and OAT are equal, F_1I must equal b, so that we have the fundamental relation between the constants of the hyperbola.

$$a^2 + b^2 = c^2. \qquad (1)$$

From any point on either branch of the hyperbola draw the focal radii PF_1 and PF_2, represented by r_1 and r_2 respectively. Then from the figure

$$r_1^2 = (x - c)^2 + y^2. \tag{2}$$

But from the equation of the hyperbola

$$y^2 = \left(\frac{b^2}{a^2}\right)(x^2 - a^2), \tag{3}$$

hence

$$r_1^2 = (x - c)^2 + \frac{b^2(x^2 - a^2)}{a^2} \tag{4}$$

$$= \frac{(a^2x^2 - 2a^2cx + a^2c^2 + b^2x^2 - a^2b^2)}{a^2} \tag{5}$$

$$= \frac{(c^2x^2 - 2a^2cx + a^4)}{a^2} \tag{6}$$

$$= \frac{(cx - a^2)^2}{a^2}. \tag{7}$$

Hence

$$r_1 = \left(\frac{c}{a}\right)x - a. \tag{8}$$

In like manner it may be shown that

$$r_2 = \left(\frac{c}{a}\right)x + a. \tag{9}$$

Hence from Eqs. (8) and (9) it follows

$$\mathbf{r_2 - r_1 = 2a.} \tag{10}$$

Thus *in any hyperbola the difference between the distances of any point on it from the foci is constant and equal to the transverse axis.*

The above relation may be derived in terms of the parametric angle ϕ. Since in any hyperbola $x = a \sec \phi$ and $y = b \tan \phi$,

$$r_1^2 = b^2 \tan^2 \phi + (a \sec \phi - c)^2$$

$$= b^2 \tan^2 \phi + a^2 \sec^2 \phi - 2ac \sec \phi + c^2.$$

To put the right-hand side in the form of a perfect square, write $b^2 = c^2 - a^2$. Then

$$r_1^2 = c^2 \sec^2 \phi - 2ac \sec \phi + a^2.$$

Therefore

$$\mathbf{r_1 = c \sec \phi - a} \tag{11}$$

and

$$\mathbf{r_2 = c \sec \phi + a.} \tag{12}$$

211. The Ratio Property of the Hyperbola.—Through the points of intersection of the a-circle with the asymptotes, draw IK, and $I'K'$. These lines are called the **directrices** of the hyperbola. It will now be proved that the ratio of the distance of any point of the hyperbola from a focus to its distance from the corresponding directrix is constant. Adopt the notation

$$\frac{c}{a} = \sec \beta = e. \tag{1}$$

Then, from the figure

$$\frac{PF_1}{PH} = \frac{r_1}{(x - ON)} = \frac{r_1}{(a \sec \phi - a \cos \beta)}. \tag{2}$$

Substituting r_1 from Eq. (11) above:

$$\frac{PF_1}{PH} = \frac{c \sec \phi - a}{a \sec \phi - a \cos \beta} \tag{3}$$

$$= \frac{c \sec \phi - a}{a\left(\sec \phi - \frac{a}{c}\right)} = \frac{c}{a}, \tag{4}$$

which proves the theorem. The constant ratio e is called the **eccentricity** of the hyperbola, and, as shown by Eq. (1), is always greater than unity.

Assuming the converse of the above, it is obvious that the hyperbola might have been defined as followed: *The hyperbola is the locus of a point whose distance from a fixed point* (*called the focus*) *is in a constant ratio, greater than unity, to its distance from a fixed line* (*called the directrix*). Proof will be given in Sec. **213.**

212. The Latus Rectum.—The double ordinate through the focus is called the **latus rectum** of the hyperbola. The value of the semi-latus rectum is readily found from the equation

$$y = \left(\frac{b}{a}\right)\sqrt{x^2 - a^2}$$

by substituting c for x. If l represent the corresponding value of y,

$$l = \left(\frac{b}{a}\right)\sqrt{c^2 - a^2} = \frac{b^2}{a}. \tag{1}$$

Hence the entire latus rectum is represented by

$$2\,l = \frac{2b^2}{a}. \tag{2}$$

Equation (1) may also be written

$$l = b\sqrt{\frac{c^2}{a^2} - 1}$$

$$l = b\sqrt{e^2 - 1}. \tag{3}$$

In Fig. 165 the distances AF_1, AN, ON, OB, OF_1, F_1N, may readily be expressed in terms of a and e, as follows in Eqs. (4) to (8). Collecting in a single table the other important formulas for the hyperbola, we have:

$$\mathbf{AF_1} = c - a = \mathbf{a(e - 1)}. \tag{4}$$

$$\mathbf{AN} = \frac{AF_1}{e} = \frac{\mathbf{a(e - 1)}}{\mathbf{e}}. \tag{5}$$

$$\mathbf{ON} = \mathbf{a}\cos\beta = \frac{\mathbf{a}}{\mathbf{e}}. \tag{6}$$

$$\mathbf{e} = \sec\beta. \tag{7}$$

$$\mathbf{OB} = \mathbf{b} = \mathbf{a}\tan\beta = \mathbf{a}\sqrt{\mathbf{e^2 - 1}}. \tag{8}$$

$$\mathbf{OF_1} = \mathbf{c} = \mathbf{ae}.$$

$$\mathbf{F_1N} = c - ON = \mathbf{ae} - \frac{\mathbf{a}}{\mathbf{e}} = \frac{\mathbf{a(e^2 - 1)}}{\mathbf{e}}. \tag{9}$$

$$\mathbf{l} = \frac{\mathbf{b^2}}{\mathbf{a}} = \mathbf{b}\sqrt{\mathbf{e^2 - 1}} = \mathbf{a(e^2 - 1)}. \tag{10}$$

$$\mathbf{r_1} = \mathbf{ex} - \mathbf{a} = \mathbf{x}\sec\beta - \mathbf{a}. \tag{11}$$

$$\mathbf{r_2} = \mathbf{ex} + \mathbf{a} = \mathbf{x}\sec\beta + \mathbf{a}. \tag{12}$$

The important properties of the hyperbola are similar to those of the ellipse. It is a good plan to compare them in parallel columns.

Ellipse	Hyperbola
1. Definition of foci and focal radii	1. Definition of foci and focal radii
2. $a^2 = b^2 + c^2$	2. $a^2 + b^2 = c^2$
3. $r_1 + r_2 = 2a$	3. $r_2 - r_1 = 2a$
4. Eccentricity, $e = c/a = \cos\beta$	4. Eccentricity, $e = c/a = \sec\beta$
5. Definition of directrices	5. Definition of directrices
6. The ratio property, $PF_1/PH = e$	6. The ratio property, $PF_1/PH = e$
7. The latus rectum $= 2b^2/a$	7. The latus rectum $= 2b^2/a$

Exercises

1. Find the eccentricity and the axes of $x^2/4 - y^2/16 = 1$.

2. Find the eccentricity and latus rectum of the hyperbola conjugate to the hyperbola of the preceding exercise.

3. An hyperbola has a transverse axis equal to 14 units and its asymptotes make an angle of 30° with the X-axis. Find the equation of the hyperbola.

4. Find the latus rectum and locate the foci and asymptotes of $4x^2 - 36y^2 = 144$.

5. Locate the directrices of the hyperbola of the preceding exercise.

6. In Fig. 165 show that $r_2 = GK'$ and $r_1 = GI$ and hence that $r_2 - r_1 = IK'$ or $2a$.

7. Find the equation of the hyperbola having latus rectum $4\frac{1}{2}$ and $a = 2b$.

8. The eccentricity of an hyperbola is $3/2$ and its directrices are the lines $x = 2$ and $x = -2$. Write the equation and draw the curve with its asymptotes, a-circle, b-circle, and foci.

9. Find the eccentricity and axes of $3x^2 - 5y^2 = -45$.

10. Find the eccentricity of the rectangular hyperbola.

11. Describe the shape of an hyperbola whose eccentricity is nearly unity. Describe the form of an hyperbola if the eccentricity is very large.

12. Describe the hyperbola if $b/a = 2$, but a very small.

13. Write the equation of the hyperbola if (*a*) $c = 5$, $a = 3$; (*b*) $c = 25$, $a = 24$; (*c*) $c = 17$, $b = 8$.

14. Describe the locus $(x + 1)^2/7 - (y - 3)^2/5 = 1$.

15. Find the equation of the hyperbola whose center is at the origin and whose transverse axis coincides with the X-axis and which passes through the points $(4.5, -1)$, $(6, 8)$.

213. The Polar Equation of the Ellipse and the Hyperbola.—In mechanics and astronomy the polar equations of the ellipse and the hyperbola are often required with the pole or origin at the right-hand focus in the case of the ellipse and at the left-hand focus in the case of the hyperbola. In these positions the radius vector of any point on the curve will increase with the vectorial angle when $\theta < 180°$. To obtain the polar equation of the ellipse and hyperbola, make use of the ratio property of the curves, namely: That the locus of a point whose distances from a fixed point (called the focus) is in a constant ratio e to its distances from a fixed line (called the directrix) is an ellipse if $e < 1$ or a hyperbola

if $e > 1$. In Fig. 166 let F be the fixed point, or focus, IX the fixed line, or directrix, P the moving point, and $FL = l$ the semi-latus rectum. Then the problem is to find the polar equation under the condition

$$\frac{PF}{PH} = e. \tag{1}$$

If e be understood to be unrestricted in value, the work and the result will apply equally well either to the ellipse or to the hyperbola.

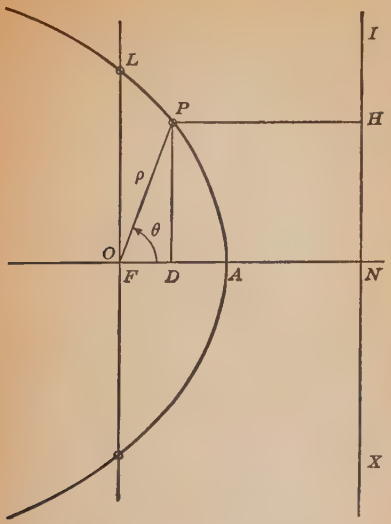

FIG. 166.—Polar equation of a conic.

When the point P occupies the position L (Fig. 166) we have $PF = l$ and $PH = FN$, whence from Eq. (1)

$$FN = \frac{l}{e}. \tag{2}$$

Take the origin of polar coordinates at F, and also take $FP = \rho$ and the angle $AFP = \theta$. Then

$$PH = FN - FD \tag{3}$$

$$FD = \rho \cos \theta. \tag{4}$$

Hence, from Eqs. (2), (3), and (4)

$$PH = \frac{l}{e} - \rho \cos \theta. \tag{5}$$

Substituting these values of FP and PH in Eq. (1), clearing of fractions, and solving for ρ, we obtain

$$\rho = \frac{l}{1 + e \cos \theta}, \tag{6}$$

which is the equation required.

When $e < 1$, Eq. (6) is the equation of an ellipse with pole at the right-hand focus. When $e > 1$, Eq. (6) is the equation of an hyperbola with the pole at the left-hand focus. In both cases the origin has been so selected that ρ increases as θ increases.

NOTE: Calling FN (Fig. 166) $= n$, Eq. (1) above may be written in rectangular coordinates

$$\frac{\sqrt{x^2 + y^2}}{n - x} = e, \tag{7}$$

or

$$x^2 + y^2 = e^2(n - x)^2, \tag{8}$$

which may be reduced to the form

$$\left(x + \frac{ne^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2 n^2}{(1 - e^2)^2}. \tag{9}$$

By Secs. **81** and **85**, this represents an ellipse if $e < 1$ or a hyperbola if $e > 1$. Thus starting with the ratio definition Eq. (7) we have proved that the curve is an ellipse or a hyperbola; that is, we have proved the statements in italics at the end of Secs. **208** and **211.**

Exercises

1. Graph on polar paper (form $M3$), the curve $\rho = \frac{6}{1 + e \cos \theta}$ for $e = 2$, $e = \frac{1}{2}$, and $e = 1$.

It will be sufficient in graphing to use $\theta = 0°$, $30°$, $60°$, $90°$, $120°$, $150°$, $180°$, $210°$, . . . , $360°$.

2. Write the polar equation of an ellipse whose semi-latus rectum is 6 ft. and whose eccentricity is $\frac{1}{3}$.

3. Write the polar equation of an ellipse whose semi-axes are 5 and 3.

4. Discuss Eq. (6) for the case $e = 0$.

5. Write the polar equation of an hyperbola if the eccentricity be $\sqrt{2}$ and the distance from focus to vertex be 4.

6. Write the polar equation of the asymptotes of

$$\rho = \frac{6}{4 + 5\cos\theta}.$$

7. Compare the curves $\rho = \dfrac{9}{4 + 5\cos\theta}$ and $\rho = \dfrac{9}{4 - 5\cos\theta}$.

8. Discuss the equation $\rho = \dfrac{l}{1 + e\cos(\theta - \alpha)}$, in which α is a constant.

214. Ratio Definition of the Parabola.—Among the curves of the parabolic type previously discussed, the one whose equation is of the second degree is of paramount importance. On that account when the term **parabola** is used without qualification, it is understood that the curve is the parabola of the second degree, whose equation may be written $y^2 = ax$ or $x^2 = ay$.

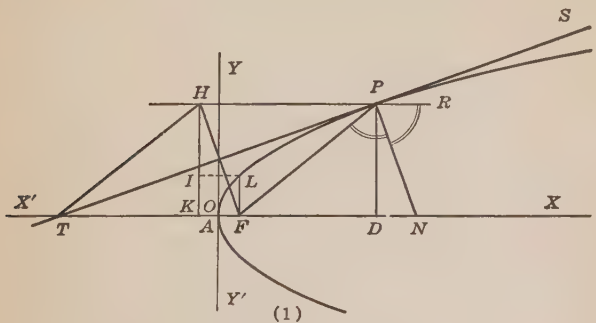

FIG. 167.—Properties of the parabola $y^2 = 4px$.

We shall prove: *The locus of a point whose distance from a fixed point is always equal to its distance from a fixed line is a parabola.* In Fig. 167, let F be the fixed point and HK the fixed line. Take the origin at A half way between F and HK. Let P be any point satisfying the condition $PF = PH$. Call $OD = x$, $PD = y$, and represent the given distance FK by $2p$. Then, from the right triangle PFD,

$$\begin{aligned} PF^2 &= y^2 + FD^2 \\ &= y^2 + (x - OF)^2 \\ &= y^2 + (x - p)^2. \end{aligned} \qquad (1)$$

Since PF by definition equals PH or $x + p$, we have

$$(x + p)^2 = y^2 + (x - p)^2, \tag{2}$$

whence

$$\mathbf{y^2 = 4px}, \tag{3}$$

which is the equation of the parabola in terms of the focal distance, OF or p.

The double ordinate through F is called the **latus rectum.**

The semi-latus rectum can be obtained from Eq. (3) by placing $x = p$, whence

$$\mathbf{l = 2p}, \tag{4}$$

where l is the semi-latus rectum. Hence the entire latus rectum is $4p$, or the coefficient of x in Eq. (3).

In Fig. 167, the quadrilateral $FLIK$ is a square, since FL and FK are each equal to $2p$.

215. Polar Equation of the Parabola.—In accordance with the ratio definition of the parabola, its polar equation is found at once from Eq. (6), Sec. **213,** by putting $e = 1$. Hence the polar equation of the parabola is

$$\rho = \frac{\mathbf{l}}{\mathbf{1 + \cos \theta}}. \tag{1}$$

For this equation we may make the following table of values:

θ	ρ
0°	$l/2$
90°	l
180°	∞
270°	l

This shows that the parabola has the position shown in Fig. 166. This is the form in which the polar equation of the parabola is used in mechanics and astronomy.

216. The Conics.—It is now obvious that a single definition can be given that will include the ellipse, hyperbola, and parabola. These curves taken together are called **the conics.** The definition may be worded: *A conic is the locus of a point whose distances from a fixed point* (*called the focus*) *and a fixed line* (*called the*

directrix) are in a constant ratio. The unity between the three curves was shown by their equation in polar coordinates. Moving the ellipse so that its left vertex passes through the origin, as in Sec. **80,** and writing the hyperbola with the origin at the right vertex (so that both curves pass through the origin in a comparable manner), we may compare each with the parabola as follows:

$$\text{The ellipse:} \quad y^2 = 2lx - \left(\frac{b^2}{a^2}\right)x^2. \tag{1}$$

$$\text{The parabola:} \quad y^2 = 2lx. \tag{2}$$

$$\text{The hyperbola:} \quad y^2 = 2lx + \left(\frac{b^2}{a^2}\right)x^2 \tag{3}$$

In these equations l stands for the semi-latus rectum of each of the curves. These equations may also be written

$$y^2 = 2lx - \left(\frac{l}{a}\right)x^2. \tag{4}$$

$$y^2 = 2lx. \tag{5}$$

$$y^2 = 2lx + \left(\frac{l}{a}\right)x^2, \tag{6}$$

whence it is seen that, if l be kept constant while a be increased without limit, the ellipse and the hyperbola each approach the parabola as near as we please. Only for large values of x, if a be large, is there a material difference in the shapes of the curves.

Exercises

1. Write the equation of the circle in the form of Eq. (1) above.

2. Write the equation of the equilateral hyperbola in the form of Eq. (3) above.

3. Describe the curve

$$\rho = \frac{l}{1 + \cos(\theta - \alpha)},$$

where α is a constant.

4. In Fig. 168 translate the curve $xy = 1$, by a suitable change in the equation, to the position shown by the dotted curve, the translation of each point being unity.

5. In Fig. 169 translate the curve $y^2 = 4px$ by a suitable change in the equation to the position shown by the dotted curve, the distance each point is moved being $3p$.

6. A bridge truss has the form of a circular segment, as shown in Fig. 170. If the total span be 80 yards and the altitude BS be 20 yards, find the ordinates C_1D_1, C_2D_2, etc. erected at uniform intervals of 10 yards along the chord AA_1.

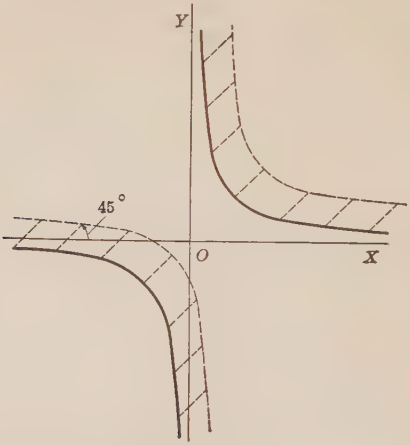

FIG. 168.—A hyperbola translated at an angle of 45° to OX.

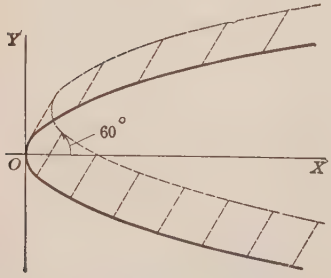

FIG. 169.—A parabola translated at an angle of 60° to OX.

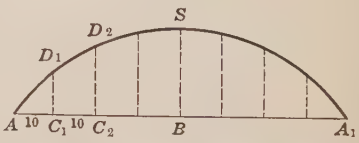

FIG. 170.—Bridge truss in form of circular segment.

7. A bridge truss has the form of a parabolic segment, as shown in Fig. 171. The span AA_1 is 24 yards and the altitude OB is 10 yards. Find the length of the ordinates CD, C_1D_1, . . . erected at uniform intervals of 3 yards along the line AA_1.

217*. The Conics Are Conic Sections.—The curves now known as the conics were originally studied by Greek geometers as the sections of a circular cone cut by a plane. It can be shown that

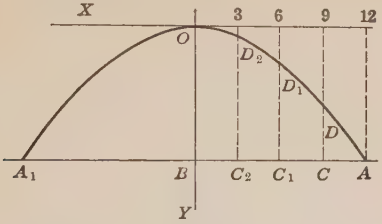

FIG. 171.—Bridge truss in the form of a parabolic segment.

the three classes of curves—parabola, ellipse, and hyperbola—can be made respectively by cutting any circular cone: (1) by a plane parallel to an element; (2) by a plane cutting opposite elements of the same nappe of the cone; (3) by a plane cutting both nappes of

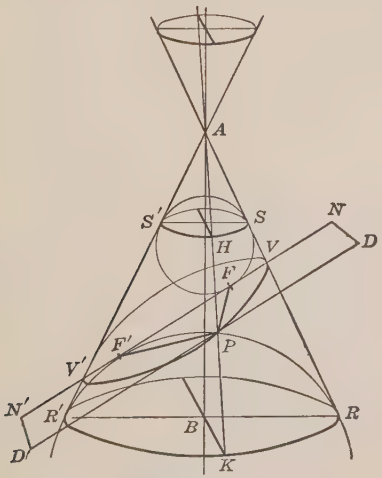

FIG. 172.—Section of a circular cone.

the cone. The two nappes of a conical surface, it will be remembered, are the two portions of the surface separated by the apex.

In Fig. 172, let the plane $NDN'D'$, called the cutting plane, cut the lower nappe of a right circular cone in the curve VPV'. It

can be proved by geometry that this curve is an ellipse. The foci F and F' are the points at which the two inscribed spheres SFS' and $RF'R'$ are tangent to the plane ND'. The directrices are the two lines ND and $N'D'$ in which the plane ND' cuts the two planes SHS' and RKR' produced.

218. Tangent to the Parabola.—Let us investigate the condition that the line $y = mx + b$ shall be tangent to the parabola $y^2 = 4px$. First find the points of intersection of these loci by solving the two equations

$$y = mx + b \tag{1}$$
$$y^2 = 4px \tag{2}$$

as simultaneous equations.

Eliminating y by substituting the value of y from Eq. (1) in Eq. (2),

$$m^2x^2 + 2mbx + b^2 - 4px = 0, \tag{3}$$

or

$$m^2x^2 + 2(mb - 2p)x + b^2 = 0. \tag{4}$$

Solving for x (see formula for quadratic, Appendix Sec. **309,** (*c*), Eq. (2)).

$$x = -\frac{mb - 2p}{m^2} \pm \frac{2\sqrt{p^2 - pmb}}{m^2}. \tag{5}$$

Therefore, there are, in general, *two* values of x or two points of intersection of the straight line and the parabola. By the definition of a tangent to a curve (Sec. **130**), this line becomes a tangent to the parabola when the two points of intersection become a single point; that is, when the expression under the radical in Eq (5) approaches zero. This condition requires that

$$p^2 - pmb = 0,$$

or

$$b = \frac{p}{m}. \tag{6}$$

Therefore, when b of Eq. (1) has this value, the line is tangent to the parabola. The equation of the tangent line is, therefore,

$$\mathbf{y = mx + \frac{p}{m}}. \tag{7}$$

This line is tangent to the parabola $y^2 = 4px$ for all values of m. Substituting in Eq. (5) the value of $b = p/m$, we may find the abscissa of the point of tangency

$$x_1 = \frac{p}{m^2}. \tag{8}$$

Substituting this value of x in Eq. (7) the corresponding ordinate of this point is found to be

$$y_1 = \frac{2p}{m}. \tag{9}$$

219. Properties of the Parabola.—In Fig. 167, F is the focus, HK is the directrix, PT is a tangent at any point P. The perpendicular PN to the tangent at the point of tangency is called the **normal** to the parabola. The projection DT of the tangent PT on the X-axis is called the **subtangent** and the projection DN of the normal PN on the X-axis is called the **subnormal.** The line through any point parallel to the axis, as PR, is known as a **diameter** of the parabola.

1. *The subtangent to the parabola at any point is bisected by the vertex.* It is to be proved that $OT = OD$ for all positions of P. Now OD is the abscissa of P, which has been found to be p/m^2. From the equation of the tangent

$$y = mx + \frac{p}{m},$$

the intercept OT on the X-axis is found by putting $y = 0$ and solving for x. This yields

$$x = -\frac{p}{m^2}.$$

This is numerically the same as OD, hence the vertex O bisects DT.

2. *The subnormal to the parabola at any point is constant and equal to the semi-latus rectum.*

The angle DPN has its sides mutually perpendicular to the sides of the angle DTP; hence the angles are equal. Since the tangent of the angle $DTP = m$, therefore

$$\text{tangent } DPN = m.$$

From the right triangle PDN,

$$DN = PD \tan DPN = PDm$$
$$= \left(\frac{2p}{m}\right) m = 2p.$$

Since KF also equals $2p$, we have

$$DN = KF.$$

3. *$PFTH$ is a rhombus.* To prove the figure $PFTH$ a rhombus it is merely necessary to show that $FT = PH$, since $PF = PH$. Now

$$FT = FO + OT$$
$$PH = DK = DO + OK.$$

But

$$OD = OT \text{ and } OK = FO.$$

Therefore

$$FT = PH,$$

and the figure is a rhombus.

It follows that the two diagonals of the rhombus intersect at right angles on the Y-axis.

4. *The normal to a parabola bisects the angle between the focal radius and the diameter at the point.* We are to show that

$$\angle NPF = \angle NPR.$$

Since $FPHT$ is a rhombus,

$$\angle FPT = \angle TPH.$$

But

$$\angle TPH = \angle RPS,$$

being vertical angles. From the two right angles NPT and NPS subtract the equal angles last named. Hence,

$$\angle \mathbf{FPN} = \angle \mathbf{NPR}.$$

It is because of this property of the parabola that the reflectors of locomotive or automobile headlights are made parabolic. The rays from a source of light at F are reflected in lines parallel to the axis, so that, in the theoretical case, a beam of light is sent out in parallel lines, or in a beam of undiminishing strength.

220. To Draw a Parabolic Arc.—One of the best ways of describing a parabolic arc is by drawing a large number of tangent

lines by the principle of Sec. **219** (3). Since in Fig. 167 the tangent is for all positions perpendicular to the focal line FH at the point where the latter crosses OY, it is merely necessary to

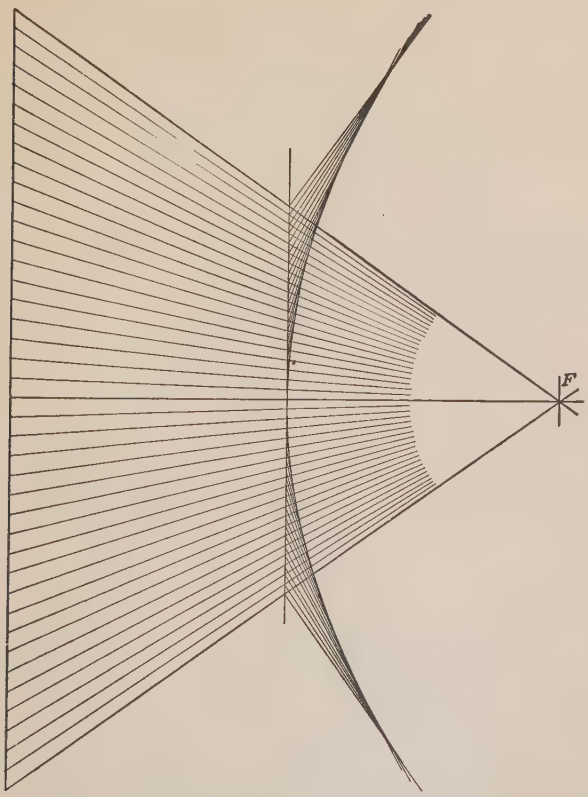

FIG. 173.—Graphical construction of a parabolic arc "by tangents."

draw a large number of focal lines, as in Fig. 173, and erect perpendiculars to them at the points where they cross the Y-axis.

The equations of the tangent lines in Fig. 173 are of the form

$$y = mx + \frac{p}{m}, \tag{1}$$

in which p is the constant given by the equation of the parabola, and in which m takes on in succession a sequence of values appropriate to the various tangent lines of the figure. These lines are said to constitute a **family of lines** and to **envelop** the curve to which they are tangent. The curve itself is called the **envelope** of the family of lines.

The curve of the supporting surface of an aeroplane as well as the curve of the propeller blades is a parabolic arc. The curve of the cables of a suspension bridge is also parabolic.

Exercises

1. Write the equation of the parabola which the family $y = mx + 7/2m$ envelops.

2. Draw an arc of a parabola if $p = 3$ in.

3. At what point is $y = mx + 3/m$ tangent to the parabola $y^2 = 12x$?

4. At what point is $y = mx + 11/m$ tangent to $y^2 = 44x$?

5. Draw the family of lines $y = mx + 1/m$ for $m = 0.4$, $m = 0.6$, $m = 0.8$, $m = 1$, $m = 2$, $m = 4$, $m = 8$.

221. Tangent to the Circle.—An equation of a tangent line to a circle can be found, as in the case of the parabola above, by finding the points of intersection of

$$y = mx + b \tag{1}$$

and

$$x^2 + y^2 = a^2 \tag{2}$$

and then imposing the condition that the two points of intersection shall become a single point. The value of b that satisfies this condition when substituted in Eq. (1) gives the equation of the required tangent. It is easier, however, to obtain this result by the following method. In Fig. 174 let the straight line be drawn tangent to the circle at T. Let the slope of this line be m. Then $m = \tan ONT = \tan \alpha$, if α be the direction angle of the tangent line. The intercept b of the line on the Y-axis can be expressed in terms of a and α,

$$b = a \sec \alpha = a\sqrt{1 + m^2}. \tag{3}$$

Hence the equation of the tangent to the circle is

$$\mathbf{y = mx \pm a\sqrt{1 + m^2}.}$$

The double sign is written in order to include in a single equation the two tangents of given slope m, as illustrated in the figure.

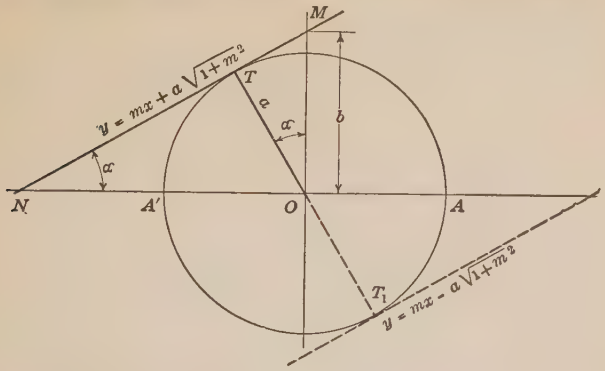

FIG. 174.—The equation of a line of given slope, tangent to a given circle.

Exercises

1. Find the equations of the tangents to $x^2 + y^2 = 16$ making an angle of 60° with the X-axis.

2. Find the equations of the tangents to $x^2 + y^2 = 25$ making an angle of 45° with the X-axis.

3. Find the equation of tangents to $x^2 + y^2 = 25$ parallel to $y = 3x - 2$.

4. Find the equation of tangents to $x^2 + y^2 = 16$ perpendicular to $y = (\frac{1}{2})x + 3$.

5. Find the equations of the tangents to $(x - 3)^2 + (y - 4)^2 = 25$ whose slope is 3.

6. Find the equation of the tangent to the circle by the method of Sec. **218.**

222. Tangent to a Circle at a Given Point.—The equation of a line having a given slope m and tangent to a given circle with center at the origin was given in Sec. **221.** We shall now find the equation of the line that is tangent to the circle *at a given point* (x_0, y_0).

The line,

$$a = \rho \cos (\theta - \alpha), \tag{1}$$

or its equivalent,

$$x \cos \alpha + y \sin \alpha = a, \tag{2}$$

is tangent to the circle of radius a, and the point of tangency is at the end of the diameter whose direction angle is α. The point of tangency is therefore $(a \cos \alpha, a \sin \alpha)$. Hence, multiplying Eq. (2) through by a, we obtain

$$x(a \cos \alpha) + y(a \sin \alpha) = a^2, \tag{3}$$

or

$$\mathbf{x_0x + y_0y = a^2}. \tag{4}$$

This is the equation of the line tangent to the circle of radius a at the point (x_0, y_0).

Thus $3x + 4y = 25$ is tangent to $x^2 + y^2 = 25$ at $(3, 4)$.

223. Tangent to the Ellipse at a Given Point.—It is easy to draw the tangent to the ellipse at any desired point. In Fig. 175 let P_0 be the point at which a tangent is desired. Draw the

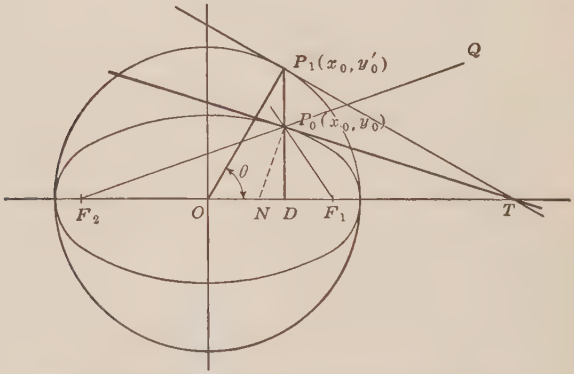

FIG. 175.—Tangent to the ellipse at a given point.

major circle, and let P_1 of the circle be a point on the same ordinate as P_0. Draw a tangent to the circle at P_1 and let it meet the X-axis at T. Then when the circle is projected to form the ellipse, the straight line P_1T is projected to make the tangent to the ellipse. Since T, when projected, remains the same point and since P_0 is the projection of P_1, the line through P_0 and T is the required tangent to the ellipse.

The equation of the tangent P_0T is also readily found. The equation of P_1T is

$$xx_0 + yy_0' = a^2. \tag{1}$$

To project this into the line P_0T, it is merely necessary to multiply the ordinates y and y_0' by b/a; that is, to substitute $y = ay/b$ and $y_0' = ay_0/b$. Whence Eq. (1) becomes

$$x_0x + \frac{a^2y_0y}{b^2} = a^2 \tag{2}$$

or, dividing by a^2,

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1, \tag{3}$$

which is the tangent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

Exercises

1. Find the equations of the tangents to the ellipse whose semi-axes are 4 and 3 at the points for which $x = 2$.

2. Find the equations of the tangents to $x^2/16 + y^2/9 = 1$ at the ends of the left-hand latus rectum.

3. Required the tangents to $x^2/9 + y^2/4 = 1$, making an angle of 45° with the X-axis. (Solve $y = x + b$ and $x^2/9 + y^2/4 = 1$ as in Sec. **218.**)

4. Find the equations of the tangents to $x^2/100 + y^2/25 = 1$ at the points where $y = 3$.

5. Find the equations of the tangents to $x^2/36 + y^2/16 = 1$ at the points where $x = y$.

224. The Tangent, Normal, and Focal Radii of the Ellipse.—In the right triangle P_1OT (Fig. 175) the side P_1O is a mean proportional between the entire hypotenuse OT and the adjacent segment OD. That is,

$$a^2 = x_0 \times OT, \text{ or } OT = \frac{a^2}{x_0}.$$

Then

$$F_1T = OT - OF_1 = OT - ae$$
$$= \frac{a^2}{x_0} - ae.$$

Likewise,

$$F_2T = OT + OF_2$$
$$= \frac{a^2}{x_0} + ae.$$

Therefore

$$\frac{F_1T}{F_2T} = \left(\frac{a^2 - aex_0}{x_0}\right)\left(\frac{x_0}{a^2 + aex_0}\right)$$
$$= \frac{(a - ex_0)}{(a + ex_0)}.$$

But by Sec. **209** this last ratio is equal to r_1/r_2. Therefore, we may write $F_1T/F_2T = P_0F_1/P_0F_2$.

Hence T, which divides the base F_2F_1 of the triangle $P_0F_2F_1$ externally at T in the ratio of the two sides PF_2 and PF_1 of the triangle, lies on the bisector of the external angle F_1P_0Q of the triangle $F_2P_0F_1$. This proves the important theorem:

The tangent to the ellipse bisects the external angle between the focal radii at the point.

This theorem provides a second method of constructing a tangent at a given point of an ellipse, often more convenient than that of Sec. **223,** since the method of Sec. **223** often runs the construction off of the paper.

The normal P_0N, being perpendicular to the tangent, must bisect the internal angle $F_2P_0F_1$ between the focal radii F_2P_0 and F_1P_0.

Since the angle of reflection equals the angle of incidence for light, sound, and other wave motions, a source of light or sound at F_1 is "brought to a focus" again at F_2, because the normal to the ellipse bisects the angle between the focal radii.

225. The Circle through Three Given Points.—In general, the equation of a circle can be found when three points are given. Either of the general equations of the circle,

$$(x - h)^2 + (y - k)^2 = a^2, \tag{1}$$

or

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{2}$$

contains three unknown constants, so that, in general, three conditions may be imposed upon them. It is best to illustrate the

general method by a particular example. Let the three given points be $(-1, 3)$, $(0, 2)$, and $(5, 0)$. Then since the coordinates of these points must satisfy the equation of the circle, we obtain from Eq. (2) above

$$1 + 9 - 2g + 6f + c = 0, \tag{3}$$

$$4 + 4f + c = 0, \tag{4}$$

$$25 + 10g + c = 0. \tag{5}$$

Eliminating c from Eqs. (3) and (4) and from (4) and (5), we obtain

$$6 - 2g + 2f = 0,$$

$$21 + 10g - 4f = 0.$$

Eliminating f

$$g = -5\tfrac{1}{2}.$$

Whence

$$f = -8\tfrac{1}{2}.$$

and

$$c = 30.$$

So the equation of the circle is

$$x^2 + y^2 - 11x - 17y + 30 = 0.$$

Exercises

1. Find the equation of the circle passing through (2, 8), (5, 7), and (6, 6).

2. Find the equation of the circle passing through (1, 2), $(-2, 3)$, and $(-1, -1)$.

226. Ellipse with Major Axis at 45° to the *OX*-axis.—The ellipse frequently arises in applied science as the resultant of the projection of the motion of two points moving uniformly on two circles, as has already been explained in Sec. **164.** Thus the parametric equations,

$$x = a \cos t \tag{1}$$

$$y = b \sin t, \tag{2}$$

define an ellipse which may be considered the resultant of two S.H.M.'s in quadrature. We shall prove that, when α is less than 180°, the equations,

$$x = a \cos t \tag{3}$$

$$y = a \sin (t + \alpha), \tag{4}$$

define an ellipse, with major axis making an angle of 45° with OX.

The graph is readily constructed as in Fig. 176. The Cartesian equation of the curve is found by eliminating t between Eqs. (3) and (4). Expanding the $\sin(t + \alpha)$ in Eq. (4) and substituting from Eq. (3) we obtain

$$y = x \sin \alpha + \sqrt{a^2 - x^2} \cos \alpha. \tag{5}$$

Transposing and squaring,

$$x^2 - 2xy \sin \alpha + y^2 = a^2 \cos^2 \alpha. \tag{6}$$

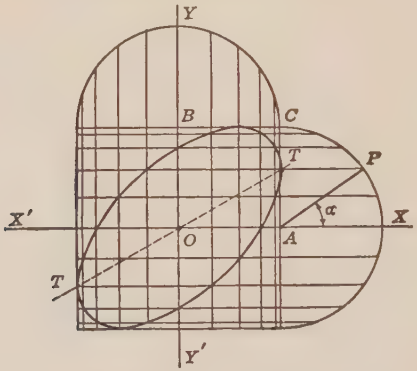

FIG. 176.—The ellipse $x = a \cos t$, $y = a \sin(t + \alpha)$.

By Sec. **200** rotate the curve through an angle of $-45°$. We know that $x^2 + y^2$ is unchanged and that $2xy$ is to be replaced by $x^2 - y^2$. Therefore Eq. (6) becomes

$$x^2(1 - \sin \alpha) + y^2(1 + \sin \alpha) = a^2 \cos^2 \alpha. \tag{7}$$

Replacing $\cos^2 \alpha$ by $1 - \sin^2 \alpha$, and dividing through by the right-hand member, we obtain

$$\frac{x^2}{a^2(1 + \sin \alpha)} + \frac{y^2}{a^2(1 - \sin \alpha)} = 1, \tag{8}$$

which may be written

$$\frac{x^2}{2a^2 \cos^2 \beta/2} + \frac{y^2}{2a^2 \sin^2 \beta/2} = 1, \tag{9}$$

where β is the complement of α. Equation (8) or (9) proves that the locus is an ellipse. It is *any* ellipse, since by properly choosing a and α the denominators in Eq. (8) can be given any

desired values. Hence the pair of parametric equations (Eqs. (3) and (4)), or the Cartesian equation (Eq. (5)) represents an ellipse with its major axis inclined $+45°$ to the X-axis.

227. Shear of the Circle.—The effect of the addition of the term mx to $f(x)$, in the equation $y = f(x)$, has been shown in Sec. **38** to change the shape of the locus by lamellar, or shearing, motion in the XY-plane. We usually speak of this process as "the shear of the locus $y = f(x)$ in the line $y = mx$." When applied to the circle $y = \pm\sqrt{a^2 - x^2}$ the effect is to move vertically the middle point of each double ordinate of the circle to a position on the line $y = mx$. The result of the shearing motion is shown in Fig. 177. *The area bounded by the curve is unchanged by the shear.*

The equation after shear is

$$y = mx \pm \sqrt{a^2 - x^2}. \tag{1}$$

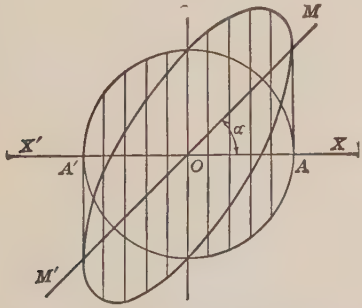

Fig. 177.—The ellipse looked upon as the shear of the circle OA in a line $M'OM$.

This is the same form as Eq. (5) of Sec. **226,** if we put $m = \sin\alpha/\cos\alpha$ and replace $y \cos\alpha$ by y_1. After the substitution, rotate the curve through 45°, and replace y_1 by $y \cos\alpha$. The equation can then be written

$$\frac{x^2}{a^2(1 + \sin\alpha)} + \frac{y^2(1 + \sin\alpha)}{a^2} = 1. \tag{2}$$

Therefore the curve of Fig. 177 is an ellipse.

The straight line $y = mx$ passes through the middle points of the parallel vertical chords of the ellipse

$$y = mx + \sqrt{a^2 - x^2}. \tag{3}$$

The locus of the middle points of parallel chords of any curve is called a **diameter** of that curve. We have thus shown that one diameter of the ellipse is a straight line. Since the same reasoning applies to

$$y = mx + \left(\frac{b}{a}\right)\sqrt{a^2 - x^2}, \tag{4}$$

which may be regarded as any ellipse in any way oriented with respect to the origin, the proof shows that the midpoints of arbitrarily selected parallel chords of an ellipse is always a straight line.

228. General Equation of the Second Degree.—The general equation of the second degree in two variables may be written in the standard form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}$$

In treatises on Conic Sections it is shown that the *general equation of the second degree in two variables represents a conic*. Three cases are distinguished as follows:

The general equation of the second degree represents

an ellipse if $h^2 - ab < 0$ (2)

a parabola if $h^2 - ab = 0$ (3)

an hyperbola if $h^2 - ab > 0$. (4)

To render the above classification true in all cases, we must classify the "imaginary ellipse," $x^2/a^2 + y^2/b^2 = -1$, as an ellipse, and other degenerate cases must be similarly treated. The expression $h^2 - ab$ is called the **quadratic invariant** of Eq. (1), so called because its value remains unchanged as the curve is moved about in the coordinate plane. In other words, as the locus of Eq. (1) is translated or rotated to any new position in the plane, and while, of course, the coefficients of x^2, xy, and y^2 change to new values, the function of these coefficients, $h^2 - ab$, does *not* change in value, but remains invariant. The above facts are not proved in this book.

229. Confocal Conics.—Figure 178 shows a number of ellipses and hyperbolas possessing the same foci A and B. This family of curves may be represented by the single equation,

$$\frac{x^2}{a^2 - k} + \frac{y^2}{b^2 - k} = 1, \tag{1}$$

in which the parameter k takes on any value less than a^2, and in which $a > b$. If k satisfies the inequality

$$k < b^2,$$

the curves are ellipses. If k satisfies the inequalities

$$b^2 < k < a^2,$$

the curves are hyperbolas. The ellipses of Fig. 178 may be regarded as representing the successive positions of the wave front of sound waves leaving the sounding body AB, or they may be

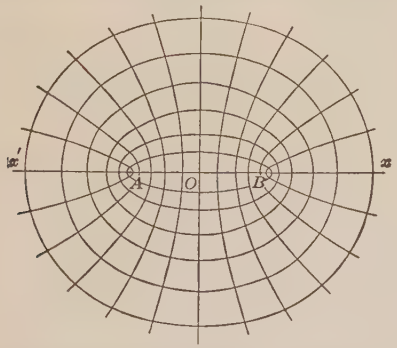

FIG. 178.—Confocal ellipses and hyperbolas. Note that the curves of one set cut the curves of the other set *orthogonally*.

regarded as the equipotential lines around the magnet AB, of which the hyperbolas represent the lines of magnetic force.

Exercises

1. Sketch the curve $y = 2x + \sqrt{4 - x^2}$.
2. Draw the curve

$$x = 2 \cos \phi.$$
$$y = 2 \sin \left(\phi + \frac{\pi}{6}\right).$$

3. Find the axes of the ellipse

$$x = 3 \cos \phi.$$
$$y = 3 \sin \left(\phi + \frac{\pi}{4}\right).$$

4. Draw the curve $y = x \pm \sqrt{6x - x^2}$.
5. Draw the curve $y = x \pm \sqrt{x^2 - 6x}$.

6. Show that $y = x \pm \sqrt{6x}$ is a parabola.

7. Sketch the curve $y = (\frac{1}{2})x + \sqrt{16 - x^2}$.

8. Sketch the curve $y = 5x \sin 60° + \cos 60° \sqrt{25 - x^2}$.

9. Discuss the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\left(\frac{xy}{ab}\right) \sin \alpha = \cos^2 \alpha.$$

Show that the locus is always tangent to the rectangle $x = \pm a$, $y = \pm b$, and that the points of contact form a parallelogram of constant perimeter $4\sqrt{a^2 + b^2}$ for all values of α. *Hint:* Compare with Eq. (6), Sec. **226.**

10. Show that $x = a \cos(\phi - \alpha)$, $y = b \sin(\phi - \alpha)$ represents an ellipse for all values of α.

11. Prove from Eq. (8), Sec. **226,** that the distance from the end of the minor to the end of the major axis of the resulting ellipse remains the same independently of the magnitude of α.

Miscellaneous Exercises

1. Show that $\sec^2 \alpha(1 + \sec 2\alpha) \equiv 2 \sec 2\alpha$.

2. Show that $\dfrac{\sin \alpha + \sin 2\alpha}{1 + \cos \alpha + \cos 2\alpha} \equiv \tan \alpha$.

3. Show that $\dfrac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} - \dfrac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \equiv 2 \tan 2\alpha$.

4. Show that $\dfrac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} \equiv \dfrac{1 + \tan \alpha \tan \beta}{1 - \tan \alpha \tan \beta}$.

5. Show that $\dfrac{1 + \tan^2 \alpha/2}{1 - \tan^2 \alpha/2} \equiv \sec \alpha$.

6. Solve the equation $\sin^2 \alpha - 2 \cos \alpha + \frac{1}{4} = 0$.

7. Simplify the product

$$(x - 2 - \sqrt{3})(x - 2 - i\sqrt{3})(x - 2 + \sqrt{3})(x - 2 + i\sqrt{3}).$$

8. Express in the form $c \cos(a - b)$ the binomial

$$30 \cos a + 40 \sin a.$$

9. Find $\tan \theta$ by means of the formula for $\tan(A + B)$, if $\theta = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$.

10. Find $\sin \theta$, if $\theta = \sin^{-1} \frac{1}{5} + \sin^{-1} \frac{1}{7}$.

11. Find the equation of a circle whose center is the origin and which passes through the point (14, 17).

12. Discuss the curve

$$x = a\phi.$$
$$y = a(1 - \cos \phi).$$

13. Graph on polar paper $\rho^2 = a^2 \cos 2\theta$.

14. A fixed point located on one leg of a carpenter's "square" traces a curve as the square is moved, the two arms of the square, however, always passing through two fixed points A and B. Find the equation of the curve.

15. Find the parametric equations of the oval traced by a point attached to the connecting rod of a steam engine.

16. Prove that

$$\tan (45° + \tau) - \tan (45° - \tau) = \frac{4 \tan \tau}{1 - \tan^2 \tau}.$$

17. Find the quotient of $(6 - 2i)$ by $(3 + 75i)$.

18. Solve for y by inspection:

$$\sin (90° + \tfrac{1}{2}y) \cos (90° - \tfrac{1}{2}y) + \cos (90° + \tfrac{1}{2}y) \sin (90° - \tfrac{1}{2}y) = \sin y.$$

19. Write the parametric equations for the circle, the ellipse, and the hyperbola.

20. The length of the shadow cast by a tower varies inversely as the tangent of the angle of elevation of the sun. Graph the length of the shadow for various elevations of the sun.

21. From your knowledge of the equations of the straight line and circle, graph $y = ax + \sqrt{a^2 - x^2}$. (See Shearing Motion, Sec. **37.**)

22. In the same manner, sketch $y = a + x + \sqrt{a^2 - x^2}$.

23. Graph the curve $y = 1/x + x^2$. Has this curve a minimum point?

24. Find by use of logarithmic paper the equations of the curves of Fig. 179. These curves give the amounts in cents per kilowatt-hour that must be added to the price of electric power to meet fixed charges of certain given annual amounts for various load factors.

25. The angle of elevation of a mountain top seen from a certain point is 29° 4′. The angle of depression of the image of the mountain top seen in a lake 230 ft. below the observer is 31° 20′. Find the height and the horizontal distance of the mountain top, and produce a single formula for the solution of the problem.

26. Find the points of intersection of the curves

$$x^2 + y^2 = 4$$
$$y^2 = 4x.$$

27. Solve $110x^{-4} + 1 = 21x^{-2}$.

28. Solve $3(x - 7)(x - 1)(x - 2) = (x + 2)(x - 7)(x + 3)$.

29. Solve $\sin x \cos x = \frac{1}{4}$.

30. By means of a progression, show how to find the compound interest on \$1000 for 25 years at 5 per cent.

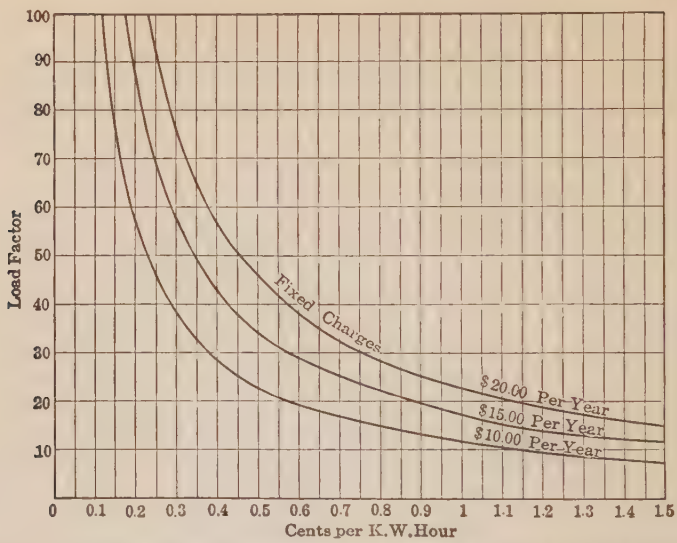

FIG. 179.—Annual fixed charges of \$10, \$15 and \$20 of a certain hydro-electric plant, reduced to cents per kw.-hr. for various load factors.

31. Find the approximate equations for the following data (two problems):

(*a*) Steam pressure: v = volume, p = pressure.

(*b*) Gas-engine mixture: v = volume, p = pressure

(*a*)		(*b*)	
v	p	v	p
2	68.7	3.54	141.3
4	31.3	4.13	115.0
6	19.8	4.73	95.0
8	14.3	5.35	81.4
10	11.3	5.94	71.2
		6.55	63.5
		7.14	54.6

32. Show that $\rho^2 = a^2 \cos 2\theta$ is the polar equation of a lemniscate.

33. By constructing the polar curves $\rho = \cos\theta$ and $\rho = \theta$, obtain a root of the equation $\cos\theta - \theta = 0$. Does the intersection of two polar curves always furnish a simultaneous solution of the equations of the two curves?

34. Write four other equations for the circle

$$\rho = 2\sqrt{3}\sin\theta - 2\cos\theta.$$

35. Write four other equations for the sinusoid

$$y = \sin x - \sqrt{3}\cos x.$$

36. Find the angle that $3x + 4y = 12$ makes with $4x - 3y = 12$.

37. From the equation

$$\theta = 6\sin(2t - 1°)$$

determine the amplitude, period, and frequency of the S.H.M.

38. A simple sinusoidal wave has a height of 3 ft., a length of 29 ft., and a velocity of 7 ft. per minute. Another wave with the same height, length, and velocity lags 15 ft. behind it. Give the equation of each.

39. Simplify

$$\frac{(3\sqrt{3} - 3i)^2(-1 + \sqrt{3}i)^3}{(2 + 2\sqrt{3}i)} + \frac{(\cos 36° + i\sin 36°)(\cos 20° + i\sin 20°)}{2(\cos 11° + i\sin 11°)}.$$

40. Calculate $(1 - \sqrt{3}i)^{\frac{2}{3}}$.

41. Plot the amount of tin required to make a tomato can to hold 1 qt. as a function of the radius of its base. Determine approximately from the graph the dimensions requiring the least tin.

42. Find the axes of the ellipse whose foci are (2, 0) and (−2, 0), and whose directrices are $x = \pm 5$.

43. Write the polar equation for the ellipse in Exercise 42.

44. Find the equation of the hyperbola whose foci are (5, 0) and (−5, 0), and whose directrices are $x = \pm 2$.

45. Write the equation of the hyperbola of Exercise 44 in polar coordinates.

46. Discuss the curve $\rho(1 + \cos\theta) = 4$. Write its equation in rectangular coordinates.

47. Find the foci of the hyperbola $2xy = a^2$. Also its eccentricity.

48. Find the equation of the locus of a point whose distance from the point (3, 4) is always twice its distance from the line $3x + 4y = 12$. What is the locus?

49. A point moves so that the quotient of its distance from two fixed points is a constant. Find the equation of the locus of the point.

50. Evaluate $\log 10 - \log_2 8 + \log_7 49^2$.

51. Find the maximum and the minimum value of $3\sin x - 4\cos x$. What values of x give these maximum and minimum values?

52. Find the equation of a circle passing through the points (1, 2), (−1, 3), and (3, −2).

53. A sinusoidal wave has a wave length of π, a period of π, and an amplitude of π. Write its equation.

54. Compute the value of each of the following:

$$\frac{3+7i}{7+3i};\ 7 \operatorname{cis} 47^\circ \times 6 \operatorname{cis} (-14^\circ);\ (7+6i)^{27};\ \sqrt[15]{7i+31}.$$

55. Prove by addition formulas that:

$$\sin (90^\circ - \tau) = \cos \tau, \quad \sin (360^\circ - \tau) = -\sin \tau,$$
$$\sin (90^\circ + \tau) = \cos \tau, \quad \tan (\tau + 270^\circ) = -\cot \tau.$$

56. Solve $x^2 + 6x + \sqrt{x^2 + 6x + 1} = 1$.

57. Find the product of $3 - 2i$ by $-2 + i$.

58. Find all the values of

$$(\cos \theta + i \sin \theta)^3;\ (\cos \theta + i \sin \theta)^{1/4};\ \sqrt[3]{1};\ \sqrt[5]{1}.$$

59. Show that

$$\sin (a+b+c) = \sin a \cos b \cos c + \cos a \sin b \cos c + \cos a \cos b \sin c - \sin a \sin b \sin c.$$

60. Draw upon squared paper, using 2 cm. = 1, the curve $y^2 = x$. By counting the small squares of the paper, find the area bounded by the curve and the ordinates $x = \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, \ldots$. By plotting these points upon some form of coordinate paper, find the functional relation existing between the x coordinate and the area under the curve.

61. The latitude of two towns is 27° 31′. They are 7 miles apart measured on the parallel of latitude. Find their difference in longitude.

62. Solve $3^{x^2-1} = 2^{x+1}$. Be very careful to take account of all questionable operations. There are two solutions.

63. Find (two problems) the equation connecting:

x	y
6.8	19.0
14.2	21.6
21.8	23.2
32.0	26.3
46.5	31.5
65.0	39.1
78.0	47.0

x	y
1.3	21
2.0	25
2.8	29
3.7	33
4.3	35
5.3	38

64. Find the wave length, period, frequency, amplitude, and velocity for $y = 10 \sin (2x - 3t)$.

65. Prove that

$$\frac{\csc^2 A}{\csc^2 A - 2} = \sec 2A.$$

66. Find the equation of the ellipse, center at the origin, axes coinciding with coordinate axes, passing through the point $(-3, 5)$ and having eccentricity $\frac{3}{5}$.

67. Prove $(\csc 2x)(1 - \cos 2x) = \sin x \sec x$.

$$(\csc x)(1 - \cos x) = ?$$

68. A S.H.M. has amplitude 6, period 3. Write its equation if time be measured from the *negative* end of the oscillation. State the difference between a S.H.M. and a wave.

69. Sketch on squared paper:

$y = 1^x$. $y = \log_2 x$
$y = 2^x$. $y = \log_3 x$
$y = 3^x$. $y = \log_5 x$.
$y = 5^x$. $y = \log_{10} x$
$y = 10^x$.

70. Solve $3^x - 2x = 1$.

71. Sketch

$\rho = a$. $\rho = \sec \theta$. $\rho = a \sin \theta$. $\rho = -a \sin \theta$.
$\rho = \frac{1}{a}$. $\rho = a \cos \theta$. $\rho = -a \cos \theta$. $\rho = a - a \cos \theta$.
$\rho = (2 - \cos \theta)$. $\rho = 2 \cos \theta - 3$. $\rho = \cos \theta + \sin \theta$.

72. Simplify the expression

$$\sin\left(\frac{\pi}{4} - \tau\right) \sec\left(\frac{\pi}{4} + \tau\right) - \sin\left(\frac{\pi}{4} + \tau\right) \sec\left(\frac{\pi}{4} - \tau\right).$$

73. Simplify and represent graphically

$$(a)\ \frac{1 + i}{1 - i};\ (b)\ (1 + i)(1 + 2i).$$

74. Find the coordinates of the center, the eccentricity, and the lengths of the semi-axes of: $(a)\ x^2 + 3x + y^2 = 7$; $(b)\ x^2 + 2x + 4y^2 - 3y = 0$; $(c)\ x^2 - x - y^2 - y = 0$; $(d)\ x^2 + x + y + 3 = 0$.

75. Find the amplitude, period, frequency, and epoch of the following S.H.M.:

$y = 7 \sin 6t$.
$y = 6 \sin 2\pi t$.
$y = a \sin (\omega t + \epsilon)$.

76. Find $\text{cis}^5\,\theta$. Show that

$$\cos 5x = \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x.$$

77. Find graphically (on form $M3$) the fifth roots of 2^5 cis $35°$.

78. Complete the following equations:

$$\sin(a \pm b) = ? \qquad \tan 2x = ?$$
$$\cos(a \pm b) = ? \qquad \cot 2x = ?$$
$$\tan(a \pm b) = ? \qquad \sin\frac{\tau}{2} = ?$$
$$\sin 2x = ? \qquad \cos\frac{\tau}{2} = ?$$
$$\cos 2x = ? \qquad \cot\frac{\tau}{2} = ?$$

79. Solve $x^3 + 1 = 0$.

80. $y = -3t^2 + 4t - 5$ and $x = 5t$ are the parametric equations of a curve. Discuss the urve.

81. Show that $[r(\cos\theta + i\sin\theta)]\,[(r'(\cos\theta' + i\sin\theta')] = rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')]$.

82. Two S.H.M.'s have amplitude 6 and period 2 ec. The point executing the first motion is $\frac{1}{4}$ sec. in advance of the point executing the second motion. Write the equations of motion.

83. Show that $\sin 5x = \sin^5 x - 10\sin^3 x\cos^2 x + 5\sin x\cos^4 x$.

84. If the thickness of panes of glass increase in arithmetical progression, the amount of light transmitted decreases in geometrical progression. Show that $L = L_0 e^{-bt}$ where t is the thickness of the glass, L is the intensity of the light transmitted, and L_0 is the initial intensity of the light. Show that if a pane of glass one unit thick ($t = 1$) absorbs 2 per cent of the incident light, then the light transmitted by ten panes of glass ($t = 10$) is $L_{10} = 100\,e^{-0.2} = 28$ per cent.

85. Newton's law of cooling states that a body surrounded by a medium of constant temperature loses heat at a rate proportional to the difference in temperature between it and the surrounding medium. If θ be the temperature of the cooling body above that of the surrounding medium at any time t, show that $\theta = \theta_0\,e^{-bt}$. What does θ_0 represent? What does the constant b represent?

86. Power is transmitted through a train of gear wheels n in number. If the loss of power in each pair of gears be 10 per cent, draw a curve showing the loss of power at the nth gear. (Use either Sec. 113, Fig. 89 or Sec. 138.)

CHAPTER XV

APPENDIX

A REVIEW OF SECONDARY SCHOOL ALGEBRA

300. Only the most important topics are included in this review From five to ten recitations should be given to this work before beginning regular work in Chapter I.

With the kind permission of Professor Hart, a number of the exercises have been taken from the *Second Course in Algebra*, by Wells and Hart.

301. Special Products. A few simple multiplications may be performed mentally.

(1) *The product of the sum and difference of any two numbers:*

$$(a + b)(a - b) = a^2 - b^2$$

From this we have $(3x - 2y)(3x + 2y) = 9x^2 - 4y^2$.

Exercises

Multiply mentally the following:

1. $(3x - y)(3x + y)$.

2. $(2x + 7)(2x - 7)$.

3. $(5x - y)(5x + y)$.

4. $(x^2y - 3a)(x^2y + 3a)$.

5. $(a + 3b)(a - 3b)$.

6. (29)(31), or $(30 - 1)(30 + 1) =$ $900 - 1 = 899$.

7. (51)(49).

8. (52)(48), or $(50 + 2)(50 - 2)$.

9. (103)(97).

10. (25)(35).

(2) *A few products of binomials are:*

$$(a + b)^2 = a^2 + 2ab + b^2.$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$
$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

Thus $(3 - a)^3 = 27 - 27a + 9a^2 - a^3.$

and $(x + y^2)^4 = x^4 + 4x^3y^2 + 6x^2y^4 + 4xy^6 + y^8.$

Expand mentally the following:

1. $(2a - x)^2$.
2. $(x + 3y)^2$.
3. $(2x - 1)^3$.
4. $(x - 3)^4$.
5. $(1 - x)^3$.
6. $(2 + y)^4$.
7. $(52)^2$, or $(50 + 2)^2$, or $2500 + 200 + 4 = 2704$.
8. $(31)^2$, or $(30 + 1)^2$.
9. $(29)^2$, or $(30 - 1)^2$.

(3) *The square of a polynomial is illustrated by the following:*

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

$$(3 - x + y)^2 = 9 + x^2 + y^2 - 6x + 6y - 2xy.$$

Expand mentally the following:

1. $(a + b + 2)^2$.
2. $(a + b - 2)^2$.
3. $(a - b - c)^2$.
4. $(2a - x + 3)^2$.
5. $(x^2 - 2y^2 + 4)^2$.
6. $(x - 2a - 6y^2)^2$.

(4) *The product of two binomials having a common term:*

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Thus
$$\begin{aligned}(x + 5)(x - 11) &= x^2 + (5 - 11)x + 5(-11),\\ &= x^2 - 6x - 55.\end{aligned}$$
$$(x + 7)(x + 2) = x^2 + 9x + 14.$$
$$(x - 5)(x + 3) = x^2 - 2x - 15.$$
$$(x^2 - 2y)(x^2 - 3y) = x^4 - 5x^2y + 6y^2.$$

Find mentally the value of each of the following:

1. $(x + 2)(x + 3)$.
2. $(x - 2)(x + 3)$.
3. $(x - 2)(x - 3)$.
4. $(x + 2)(x - 3)$.
5. $(x^2 + 5y)(x^2 - 5y)$.
6. $(3x + 2y)(3x - 7y)$.
7. $(x^2 - 3)(x^2 - 4)$.
8. $(3xy - z)(3xy + 7z)$.
9. $(x^2y^2 - 3)(x^2y^2 - 10)$.
10. $(x - 2y)(2x - 2y)$.

(5) *The product of two general binomials:*

$$(ax + b)(cx + d) = acx^2 + (bc + ad)x + bd.$$

Thus
$$\begin{aligned}(3a - 4b)(2a + 7b) &= (3a)(2a) + (-8 + 21)ab + (-4b)(7b)\\ &= 6a^2 + 13ab - 28b^2.\end{aligned}$$

Find mentally the following products:

1. $(5x - 2y)^2$.
2. $(a + 11b)(a + 3b)$.
3. $(a - 2v)(a + 12v)$.
4. $(2m + 3)(m + 4)$.
5. $(y^2 + 4z)(y^2 + 4z)$.
6. $(3xy - 7)^2$

7. $(3u^2v - 4)(3u^2v + 4)$.
8. $(2x - 5)(x + 4)$.
9. $(2r^2 - 7)(3r^2 + 5)$.
10. $(p^2 - 3q)(p^2 + 7q)$.
11. $(a + \frac{1}{2})(a - \frac{1}{3})$.
12. $(\frac{1}{2}x + 5y)(\frac{1}{2}x - 5y)$.
13. $(u - \frac{1}{2})(u - \frac{1}{4})$.
14. $(2x + 3)(\frac{1}{2}x + 1)$.
15. $(3x^2 + 4bc)(3x^2 - 4bc)$.
16. $(y - 8)(y + 5)$.
17. $(x - \frac{3}{4})(x - \frac{3}{8})$.
18. $(1 - 6s)(3 + 2s)$.
19. $(2t - 7w^2)(3t - 4w^2)$.
20. $(\frac{2}{5}u - \frac{1}{8})(\frac{2}{5}u + \frac{1}{8})$.
21. $(3r - 7t)(5r + 2t)$.
22. $(11x^2 - 1)(12x^2 + 1)$.
23. $(z^2 - 6)(z^2 + 12)$.
24. $(x + 3y^2)(x - 2y^2)$.
25. $(5m^3 - 6s^2)(5m^3 + s^2)$
26. $(5x + \frac{1}{3})(5x - \frac{1}{3})$.
27. $(3x + 7)(x - 5)$.
28. $(4a - 3b^3)^2$.
29. $(2 - 3st)(5 + 2st)$.
30. $(a^2b + 6c)(a^2b - 13c)$.
31. $(lp + 5)(lp - 4)$.
32. $(a^3 + 7)(a^3 - 11)$.
33. $(3a + 5)(7a - 8)$.
34. $(1 + 8n)(1 - 9n)$.
35. $(2a - b^4)(2a + 3b^4)$.
36. $(12x - \frac{1}{3})(9x - \frac{1}{2})$.
37. $(20 - 16z)(3 + 2z)$.
38. $(r^3 + 16s)(r^3 - s)$.
39. $(a - 6x^2)(a + x^2)$.
40. $(4r + uv)(4r - 5uv)$.
41. $(6x^2 - \frac{1}{3})^2$.
42. $(1 + 23n)(5 - n)$.
43. $(x^4 - y^4)(x^4 + y^4)$.
44. $(5a^2 - 4b)(6a^2 - 5b)$.
45. $(x^2y + y^2x)(x^2y - y^2x)$.
46. $(\frac{1}{2}a + 10)(2a + \frac{1}{5})$.
47. $(9r + 2s)(3r - 4s)$.
48. $(12x^2 + 5)(4x^2 - 3)$.
49. $(a^2b^4 + 4x^2)^2$.
50. $(a^6 - b^6)(a^6 + b^6)$.
51. $(a + b)(a - b)(a^2 + b^2)(a^4 + b^4)$.

302. Symbols of Aggregation. If a sign of aggregation is preceded by the negative sign, change all signs within when the sign of aggregation is removed. If the sign of aggregation is preceded by the positive sign, all signs within remain the same when the sign of aggregation is removed.

$$\begin{aligned} 5x^2 - [3y^2 + \{2x^2 - (y^2 + 3x^2) + 5y^2\} - x^2] \\ &= 5x^2 - [3y^2 + \{2x^2 - y^2 - 3x^2 + 5y^2\} - x^2] \\ &= 5x^2 - [3y^2 + \{4y^2 - x^2\} - x^2] \\ &= 5x^2 - [3y^2 + 4y^2 - x^2 - x^2] \\ &= 5x^2 - [7y^2 - 2x^2] \\ &= 5x^2 - 7y^2 + 2x^2 \\ &= 7x^2 - 7y^2 \end{aligned}$$

Exercises

Simplify the following by removing the signs of aggregation:

1. $ab - 4b^2 - (2a^2 - b^2) - \{-5a^2 + 2ab - 3b^2\}$.
2. $x - \{y + z - [x - (-x - y) + z]\} + [z - (2x - y)]$.

3. $a - \{ -a - [-a - (-a - 1)]\}$.
4. $3yz - [2yz + (9z - 2yz)]$.
5. $-\{ -1 - [-1 -(-1)]\}$.
6. $5x^2 - [3y^2 + \{2x^2 - (y^2 + 3x^2) + 5y^2\} - x^2]$.
7. $ab - [4b^2 - (2a^2 - b^2) - \{ -5a^2 + 2ab - 3b^2\}]$.
8. $3y^2 - [2y^2 + (9z - 2yx)]$.

303. Factoring. Since $(a \pm b)^2 = a^2 \pm 2ab + b^2$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$x^2 - 6xy + 9y^2 = (x - 3y)^2$$

and

$$4 + 4(a + b) + (a + b)^2 = (2 + a + b)^2$$

Exercises 1

Factor the following by inspection:

1. $9x^2 - 30xy + 25y^2$.
2. $4 + 16t + 16t^2$.
3. $x^4y^4 + 10x^2y^2z^2 + 25z^4$.
4. $9 + 6(x^2 + y^2) + (x^2 + y^2)^2$.
5. $a^4 + 4a^2b^2 + 4b^4$.

Since $(a + b)(a - b) = a^2 - b^2$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$4a^2 - 9b^2 = (2a + 3b)(2a - 3b).$$

Exercises 2

Factor the following by inspection:

1. $x^2y^2 - z^2$.
2. $(a + b)^2 - c^2$
3. $c^2 - (a + b)^2$.
4. $25 - 9x^2$.
5. $81 - 625x^4$.

Since $(a + b)(a + c) = a^2 + (b + c)a + bc$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$x^2 - 5x - 14 = (x - 7)(x + 2)$$

Exercises 3

Factor the following by inspection:

1. $x^2 + 7x + 10$.
2. $a^2 + 4ay - 21y^2$.
3. $4x^2 - 18xy + 18y^2$.
4. $9x^2 - 18x - 27$.
5. $25 + 30a - 27a^2$.

Since $(a + b)(a^2 - ab + b^2) = a^3 + b^3$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$27 + 125x^3 = (3 + 5x)(9 - 15x + 25x^2).$$

Exercises 4

Factor the following by inspection:

1. $x^3y^3 + 1$.
2. $x^6 + y^6$.
3. $8 + 27x^3$.
4. $125 + x^3y^3$.
5. $x^3 + 8y^6$.

Since $(a - b)(a^2 + ab + b^2) = a^3 - b^3$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$27 - 125x^3 = (3 - 5x)(9 + 15x + 25x^2).$$

Exercises 5

Factor the following by inspection:

1. $x^3y^3 - 1$.
2. $x^6 - y^6$, or $(x^3 + y^3)(x^3 - y^3)$.
3. $8 - 27x^3$.
4. $125 - x^3y^3$.
5. $x^3 - 8y^3$.
6. $27 - 8a^3$.

The following may be factored by grouping the terms. Thus,

$$\begin{aligned} a^3m + a^3n - m - n &= a^3(m + n) - (m + n) \\ &= (a^3 - 1)(m + n) \\ &= (a - 1)(a^2 + a + 1)(m + n). \end{aligned}$$

Exercises 6

Factor the following:

1. $ax - ay + bx - by$.
2. $x^3 + 3x^2 + 3x - 1$.
3. $ax^2 - 2axy + ay^2 + bx^2 - 2bxy + by^2$.
4. $x^5 - xy^4 - x^4y + y^5$.
5. $x^4 - x^3y - xy^3 + y^4$.

A trinomial of the form $px^2 + qx + r$, if the product of two binomials, may be factored as outlined below.
In the product

$$\begin{array}{r} ax + b \\ cx + d \\ \hline acx^2 + (bc + ad)x + bd \end{array}$$

the terms acx^2 and bd are called *end products* and bcx and adx are called *cross products*. This most important case of factoring is best learned from the consideration of actual examples.

Factor $21x^2 + 5x - 4$.

From the term $21x^2$, consider as possible first terms $7x$ and $3x$, thus $(7x\ \ )(3x\ \ )$. For factors of (-4), try 2 and 2, with unlike signs and signs so arranged that the cross product with larger absolute value shall be positive; thus $(7x - 2)(3x + 2)$. This gives middle term $8x$; incorrect. For (-4) try 4 and 1, with signs selected as before; thus, $(7x - 1)(3x + 4)$. Middle term $25x$; incorrect. Try $(7x + 4)(3x - 1)$. Middle term $5x$; correct.

Factor $24x^2 - 17xy + 3y^2$.

Try $(6x - 3y)(4x - y)$. Incorrect, since first ( ) contains factor 3 and given expression does not. Try $(6x - y)(4x - 3y)$. Middle term -22; incorrect. Try $(8x - 3y)(3x - y)$. Middle term -17 correct.

Exercises 7

Factor the following:

1. $6x^2 - 7x + 2$.
2. $3x^2 + 8x + 4$.
3. $6x^2 - x - 2$.
4. $9a^2 + 15a + 4$.
5. $6b^2 - 7b - 10$.
6. $14x^2 + 13xy - 12y^2$.
7. $8z^2 - 2yz - 21y^2$.
8. $35u^2 + uv - 6v^2$.
9. $9t^2 - 14t - 8$.
10. $12x^2 - 35xy - 3y^2$.
11. $6 - t - 15t^2$.
12. $5 + 9s - 18s^2$.
13. $24m^2 - 17mn + 3n^2$.
14. $28y^2 - yz - 2z^2$.

An expression of the from $a^4 + a^2b^2 + b^4$ may be put in the form of the difference of two squares by adding and subtracting a^2b^2. Thus

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2). \end{aligned}$$

Exercises 8

Factor the following:

1. $x^4 + x^2y^2 + y^4$.
2. $x^4 + 4x^2 + 16$.
3. $y^4 + 4y^2z^2 + 16z^4$.
4. $16 + 4u^2 + u^4$.
5. $16x^4 + 36x^2y^2 + 81y^4$.
6. $a^8 + a^4b^2 + b^4$.
7. $a^8x^4 + a^4x^2y^2 + y^4$.
8. $625x^4 + 100x^2z^4 + 16z^8$.

304. To factor a polynomial completely, first remove any monomial factor present; then factor the resulting expression by any of the type forms which apply, until prime factors have been obtained throughout. Thus,

$$(a)\quad 5a^6 - 5b^6 = 5(a^6 - b^6) = 5(a^3 - b^3)(a^3 + b^3)$$
$$= 5(a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2)$$
$$(b)\quad 42ax^2 + 10ax - 8a = 2a(21x^2 + 5x - 4)$$
$$= 2a(7x + 4)(3x - 1)$$
$$(c)\quad 20mnu^2 - 100mnu + 125mn = 5mn(4u^2 - 20u + 25)$$
$$= 5mn(2u - 5)^2.$$

Exercises

Factor the following expressions:

1. $\frac{1}{16}u^6 - \frac{9}{25}v^4$.
2. $9x^8 - 4y^6$.
3. $25x^4 - 1$.
4. $81 - 4x^2$.
5. $1 - 64a^2b^4c^6$.
6. $x^6 - y^8$.
7. $225 - a^6$.
8. $121x^2 - 144y^2$.
9. $49m^4 - 36x^2y^2z^2$.
10. $169 - \frac{9}{49}a^4x^2$.
11. $4x^2 - 20x + 25$.
12. $9a^2 + 6ab + b^2$.
13. $a^2b^2 - 17abc - 60c^2$.
14. $r^4 - 11r^2 + 30$.
15. $16b^2 + 30b + 9$.
16. $81u^2 + 180uv + 100v^2$
17. $36a^2 - 132a + 121$.
18. $x^2y^4 - 4xy^2 + 4$.
19. $a^2b^2 - 2ab - 35$.
20. $u^6 + u^3 - 110$.
21. $a^4b^2 - 14a^2b + 49$.
22. $x^2 + 6x - 27$.
23. $c^3 - 64t^3$.
24. $8x^3 - 1$.
25. $1 - 13t - 68t^2$.
26. $x^4 - 6x^2b - 55b^2$.
27. $au^2 - 4auv - 45av^2$.
28. $28a^2 - a - 2$.
29. $3s^2 - 17st + 24t^2$.
30. $15r^2 - r - 6$.
31. $4y^2 - 3y - 7$.
32. $64u^6 - 27x^3$.
33. $6ar - 3as + 4at$.
34. $a^2 + 2a - 35$.
35. $9x^2 + 12xy - 32y^2$.
36. $a^2 + 10ab + 25b^2$.
37. $625x^2y^2 - \frac{1}{49}$.
38. $3cdy^2 - 9cdy - 30cd$.
39. $4ax^2 - 25ay^4$.
40. $3y^3 + 24$.
41. $4x^2 - 27x + 45$.
42. $6x^2 + 7x - 3$.

43. $\frac{9}{25}z^2 - 1$.
44. $10x^3y - 5x^2y^2 - 5xy^3$.
45. $m^2n^2 + 7mn - 30$.
46. $x^2 - 3xy - 70y^2$.
47. $mx^2 + 7mx - 44m$.
48. $x^3 - 3x^2 - 108x$.
49. $x^8 - y^8$.
50. $x^4 - 5x^2y - 24y^2$.
51. $8n^2 + 18n - 5$.
52. $3x^4 - 12$.
53. $9m^2 - 42mt + 49t^2$.
54. $10x^2 - 39x + 14$.
55. $12x^2 + 11x + 2$.
56. $36x^2 + 12x - 35$
57. $x^3 - 8y^3$.
58. $2am^2 - 50a$.
59. $72 + 7x - 49x^2$.
60. $31x^2 + 23xy - 8y^2$.
61. $24a^2 + 26a - 5$.
62. $1 - 3xy - 108x^2y^2$.
63. $x^2 - 14mx + 40m^2$.
64. $2b + 10ab - 28a^2b$.
65. $c^3 + 27d^3$.
66. $3x^3y - 27xy^3$.
67. $\frac{1}{10}x^3y^2 - \frac{1}{40}x^5y^4$.
68. $49n^4y - 196n^2y^3$.
69. $x^2 - 16x + 48$.
70. $x^2 + 23x - 50$.
71. $a^4n^4 + 31a^2n^2 + 30$.
72. $9x^2 + 37xy + 4y^2$.

305. General Distributive Law in Multiplication. From the meaning of a product, we may write

$$(a + b + c + \ldots)(x + y + z + \ldots) = ax + bx + cx + \ldots + ay + by + cy + \ldots + az + bz + cz + \ldots,$$
etc.

Stating this in words: *The product of one polynomial by another is the sum of all the terms found by multiplying each term of one polynomial by each term of the other polynomial.*

To multiply several polynomials together, we continue the above process. In words we may state the generalized distributive law of the product of any number of polynomials as follows:

The product of k polynomials is the aggregate of ALL *of the possible partial products which can be made by multiplying together k terms, of which one and only one must be taken from each polynomial.*
Thus,

$$(a + b + c + \ldots)(x + y + z + \ldots)(u + v + w + \ldots)$$
$$= axu + axv + \ldots + ayu + ayv + \ldots + azu + azv + \ldots$$
$$+ bxu + bxv + \ldots + byu + byv + \ldots + bzu + bzv + \ldots$$
$$+ cxu + cxv + \ldots$$
$$+ \ldots, \text{etc.}$$

If the number of terms in the different polynomials be n, r, s, t. . . respectively, the total number of terms in the product will be nrst . . .
The student may prove this.

306. The Fundamental Theorem in the Factoring of $x^n \pm a^n$. *The expression $(x^n - a^n)$ is always divisible by $(x - a)$, when n is a positive integer.*

Write
$$x^n - a^n = x^n - ax^{n-1} + ax^{n-1} - a^n$$
$$= x^{n-1}(x - a) + a(x^{n-1} - a^{n-1})$$

Now *if* $(x^{k-1} - a^{k-1})$ is divisible by $(x - a)$, then plainly $x^{k-1}(x - a) + a(x^{k-1} - a^{k-1})$ is also divisible by $(x - a)$. But this last expression equals $(x^k - a^k)$, as we have shown. Therefore, *if* $(x - a)$ exactly divides $(x^{k-1} - a^{k-1})$, it will also exactly divide $(x^k - a^k)$.

That is, *if* the law is true for any positive integral value of k, it is true for k one greater. But by actual division the law is true when k is 3, $(x^3 - a^3 = (x^2 + ax + a^2)(x - a)$ therefore it is true when k is 4. Being true when k is 4, it is true when k is 5, and so on up to $k = n$, any positive integer.

We see that $(x - a)$ is *one* factor of $(x^n - a^n)$. The *other* factor of $(x^n - a^n)$ is found by actually dividing $(x^n - a^n)$ by $(x - a)$. Thus $(x^n - a^n) = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1})$

The student may show that $(x + a)$ divides $x^n + a^n$ if n be odd, and divides $x^n - a^n$ if n be even.

Exercises

Factor the following:

1. $x^5 + y^5$.
2. $x^5 + 32$.
3. $x^4 - 81$.
4. $x^9 + 1$.
5. $x^4 - 16y^4$.
6. $x^5y^5 + 1$.
7. $m^5 - 243$.
8. $32a^5 + 243b^5$.
9. $64 - x^6$.
10. $x^6y^6 - z^6$.
11. $x^5 - y^{10}$.
12. $27x^3 - 8y^3$.

Miscellaneous Exercises in Factoring

Factor the following:

1. $(a + b)^2 - c^2$.
2. $(m - n)^2 - x^2$.
3. $(x - y)^2 - z^2$.
4. $x^2 - (y - z)^2$.
5. $(7x - 2y)^2 - y^2$.
6. $(a + b)^2 + 23(a + b) + 60$.
7. $(x + y)^2 + 2(x + y) - 63$.
8. $(x - y)^2 - (x + y)^2$.
9. $(x^2 - 2y)^2 + 2(x^2 - 2y) + 1$.
10. $(1 + u^2)^2 - 4u^2$.
11. $9(m - n)^2 - 12(m - n) + 4$.
12. $(x^2 - 4)^2 - (x + 2)^2$.
13. $(x^2 + 3x)^2 + 4(x^2 + 3x) + 4$.
14. $(9x^2 + 4)^2 - 144x^2$.
15. $(x + y)^2 + 7(x + y) - 144$
16. $(a^2 + a + 9)^2 - 9$.
17. $(x + y)^3 - z^3$.
18. $(x + y)^3 + z^3$.

19. $x^3 - (y + z)^3$.
20. $x^3 - (y - z)^3$.
21. $x^3 + (y - z)^3$.
22. $(m + n)^3 + 8t^3$.
23. $(x + y)^3 + (x - y)^3$.
24. $27a^3 - (a - b)^3$.
25. $a^2 - 2ab + b^2 - c^2$.
26. $x^2 + 2xy + y^2 - z^2$.
27. $a^2 - x^2 - 2xy - y^2$.
28. $x^2 - y^2 - z^2 + 2yz$.
29. $b^2 - 4 + 2ab + a_2$.
30. $2mn - n^2 + 1 - m^2$.
31. $9a^2 - 24ab + 16b^2 - 4c^2$.
32. $4a^2 - 6b - 9 - b^2$.
33. $9x^4 - 4x^2 + y^2 - 6x^2y - 20xz - 25z^2$.
34. $4y^4 - 32y^2 + 1$.
35. $9t^4 - 31t^2x^2 + 25x^4$.
36. $25a^4 + 34a^2b^2 + 49b^4$.
37. $2a(x + y) - 3(x + y)$.
38. $a(x - y) - b(x - y)$.
39. $ab + an + bm + mn$.
40. $2 + 3x - 8x^2 - 12x^3$.
41. $56 - 32a + 21a^2 - 12a^3$.
42. $4a^3 + a^2b^2 - 4b^3 - 16ab$.
43. $st - sr - r^2 + rt$.
44. $x^3 + x^2 + x + 1$.

307. Fractions. Multiplying or dividing both numerator and denominator of a fraction by the same number, excepting zero, does not change the value of the fraction. To reduce a fraction to its lowest terms factor both numerator and denominator and then divide out the common factors if there are any. Thus,

$$\frac{ax^4 + ax^2y^2 + ay^4}{a^2x^3 - a^2y^3} = \frac{a(x^2 + xy + y^2)(x^2 - xy + y^2)}{a^2(x - y)(x^2 + xy + y^2)} = \frac{x^2 - xy + y^2}{a(x - y)}$$

Exercises

Reduce the following to lower terms:

1. $\dfrac{ax^2 - ay^2}{a^2x^3 - a^2y^3}$.
2. $\dfrac{x^2 + x - 6}{x^2 - 4}$.
3. $\dfrac{x^4 - y^4}{x^6 - y^6}$.
4. $\dfrac{ax + ay - x - y}{x^3 + y^3}$.
5. $\dfrac{27 - x^3}{12 - 7x + x^2}$.
6. $\dfrac{a^8 - b^8}{a^2 - b^2}$.

Miscellaneous Exercises in Fractions

Simplify the following:

1. $\dfrac{n^2 - 36}{4n^2} \cdot \dfrac{7n^2}{n^2 + n - 42}$.
2. $\dfrac{x^2 - 2x - 35}{2x^3 - 3x^2} \cdot \dfrac{4x^2 - 9x}{7(x - 7)}$.
3. $\dfrac{(5a + 2)(a - 2)}{2a^2 + a - 10}$.

4. $\dfrac{4a^2+8a+3}{2a^2-5a+3}\cdot\dfrac{6a^2-9a}{4a^2-1}$.

5. $\dfrac{16x-4}{5x-5}\cdot\dfrac{20x+5}{6x+6}\cdot\dfrac{x^2+2x+1}{16x^2-1}$.

6. $\dfrac{x^3+8y^3}{x^3-8y^3}\cdot\dfrac{x-2y}{x+2y}\cdot\dfrac{x^2+2xy+4y^2}{x^2-2xy+4y^2}$.

7. $\dfrac{2n^2-n-3}{n^4-8n^2+16}\cdot\dfrac{n^2+4n+4}{n^2+n}\cdot\dfrac{n^2-n-2}{2n^2-3n}$.

8. $\dfrac{x^2-xy-2y^2}{x^3-9xy^2}\div\dfrac{x-2y}{x-3y}$.

9. $\dfrac{2x^2-xy-3y^2}{9x^2-25y^2}\div\dfrac{3x^2+xy-2y^2}{9x^2-30xy+25y^2}$.

10. $\dfrac{2a^2-5ab-3b^2}{a^2-ab-2b^2}\div\left[\dfrac{2a^2-7ab-4b^2}{a^2-3ab-4b^2}\div\dfrac{a^2-4ab+4b^2}{a^2-ab-6b^2}\right]$.

11. $\left(\dfrac{x+2}{x}+\dfrac{x}{x-3}\right)\left(\dfrac{x}{x-2}-\dfrac{3}{x+3}\right)$.

12. $\left(2a-1+\dfrac{6a-11}{a+4}\right)\div\left(a+3-\dfrac{3a+17}{a+4}\right)$

13. $\left(x^4-\dfrac{1}{x^4}\right)\div\left((x-\dfrac{1}{x}\right)$.

14. $x+\dfrac{1}{x+\dfrac{1}{x}}$.

308. Simple Equations. Adding the same number to both members of an equation does not change the equation. It follows that a term may be transposed from one member of an equation to the other member provided its sign is changed. Thus from

$$3x-2=3+2x.$$
$$3x-2x=3+2, \text{ or } x=5.$$

Exercises

Solve the following equations for x:

1. $\dfrac{x-3}{2x+1}+\dfrac{2x-1}{4x-3}=1.$

2. $\dfrac{6x+7}{3}-\dfrac{3}{x+2}=2x+\tfrac{1}{2}.$

3. $\dfrac{x-2}{3}-\dfrac{2x+3}{1}=6.$

4. $(x+4)(x-2)=(x+3)(3x+4)-(2x+1)(x-6).$

5. $(x^2-2x+1)^2=(x-1)^2(x-3)^2.$

6. $\frac{2(5x-9)}{3} - x + 14 = \frac{3(7x-19)}{4} - 4.$

7. $\frac{4}{x-3} - \frac{3}{x-5} = \frac{1}{x}.$

8. $\frac{1}{x-8} - \frac{1}{x-6} = \frac{1}{x-4} - \frac{1}{x-2}.$

9. $\frac{x+a}{b} - \frac{b}{a} = \frac{x-b}{a} + \frac{a}{b}.$

10. $\frac{a}{x+a} - \frac{b}{x+b} = \frac{a-b}{x-b}.$

309. Quadratic equations are usually solved (*a*) by factoring, (*b*) by completing the square, or (*c*) by use of a formula.

(*a*) To solve by factoring, transpose all terms to the left member of the equation and completely factor. The solution of the equation is then deduced from the fact that if the value of a product is zero, then one of the factors must equal zero. Thus

(1) Solve the equation

$$x^2 + 54 = 15x$$

Transposing $x^2 - 15x + 54 = 0$

Factoring $(x - 9)(x - 6) = 0$

$x - 9 = 0$ if $x = 9$

$x - 6 = 0$ if $x = 6$

Hence the roots of the equation are 9 and 6

CHECK: Does $(9)^2 + 54 = 15 \times 9$?
Does $(6)^2 + 54 = 15 \times 6$?

(2) Solve the equation

$$12x^2 + x = 6$$

Transposing $12x^2 + x - 6 = 0$

Factoring $(3x - 2)(4x + 3 = 0$

$3x - 2 = 0$ if $x = \frac{2}{3}$

$4x + 3 = 0$ if $x = -\frac{3}{4}$

Hence the roots of the equation are $\frac{2}{3}$ and $-\frac{3}{4}$.

CHECK: Does $12(\frac{2}{3})^2 + \frac{2}{3} = 6$?
Does $12(-\frac{3}{4})^2 - \frac{3}{4} = 6$?

(*b*) To solve by completing the square, use the properties of $(x \pm a)^2 = x^2 \pm 2ax + a^2$, as follows:

(3) Solve $x^2 - 12x = 13$.

Add the square of 1/2 of 12 to each side

$$x^2 - 12x + 36 = 49$$

Take the square root of each member

$$x - 6 = \pm 7$$

Hence

$$x = 6 + 7 = 13$$
$$x = 6 - 7 = -1$$

CHECK: Does $(13)^2 - 12 \times 13 = 13$?
Does $(-1)^2 - 12 \times (-1) = 13$?

Since in general $(x - a)(x - b) = x^2 - (a + b)x + ab$, we can check thus:

Does $13 + (-1) = -(-12)$?
Does $13(-1) = -13$?

(4) Solve $x^2 - 20x + 97 = 0$.

Transpose 97 and add the square of 1/2 of 20 to each side:

$$x^2 - 20x + 100 = -97 + 100 = 3$$

Take the square root of each number:

$$x - 10 = \pm \sqrt{3}$$

Hence

$$x_1 = 10 + \sqrt{3}$$
$$x_2 = 10 - \sqrt{3}$$

CHECK: Does $x_1 + x_2 = -(-20)$?
Does $x_1 x_2 = 97$?

(c) To solve by use of a formula, first solve

$$ax^2 + bx + c = 0 \tag{1}$$

The roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

For a particular example, substitute the appropriate values of a, b, and c. Thus:

(5) Solve $2x^2 - 3x - 5 = 0$.

Comparing the equation term by term with (1) we have

$$a = 2,\ b = -3,\ c = -5$$

Substitute these values in the formula (2)

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$
$$= \frac{3 \pm 7}{4}$$

Therefore

$$x_1 = 5/2,\ x_2 = -1$$

CHECK: Does $x_1 + x_2 = -b/a = 3/2$?
Does $x_1 x_2 = c/a = -5/2$?

Exercises

Solve the following quadratics in any manner:

1. $x^2 + 5x + 6 = 0$.
2. $x^2 + 4x = 96$.
3. $x^2 = 110 + x$.
4. $x^2 + 5x = 0$.
5. $6x^2 + 7x + 2 = 0$.
6. $8x^2 - 10x + 3 = 0$.
7. $x^2 + mx - 2m^2 = 0$.
8. $3t^2 - t - 4 = 0$.
9. $10r^2 + 7r = 12$.
10. $x^2 + 2ax = b$.
11. $x^2 + 4x = 5$.
12. $x^2 + 6x = 16$.
13. $2x^2 - 20x = 48$.
14. $x^2 + 3x = 18$.
15. $x^2 + 5x = 36$.
16. $3x^2 + 6x = 9$.
17. $4x^2 - 4x = 8$.
18. $x^2 - 7x = -6$.
19. $x^2 - ax = 6a^2$.
20. $x^2 - 2ax = 3a^2$.
21. $x^2 - x = 2$.
22. $x^2 + x = a^2 + a$.
23. $x^2 - 10x = -9$.
24. $2x^2 - 15x = 50$.
25. $x^2 + 8x = -15$.
26. $3x^2 + 12x = 36$.
27. $2x^2 + 10x = 100$.
28. $x^2 - 5x = -4$.
29. $3x^2 - 12ax = 63a^2$.
30. $4x^2 - 12ax = 16a^2$.
31. $x^2 - x = 6$.
32. $x^2 + 7x = -12$.
33. $x^2 - 5x = 14$.
34. $x^2 + x = 12$.
35. $x^2 - x = 12$.
36. $x^2 = 6x - 5$.
37. $x^2 = -4x + 21$.
38. $x^2 = -4x + 5$.
39. $x^2 + 5x + 6 = 0$.
40. $x^2 + 11x = -30$.
41. $x^2 - 7x + 12 = 0$.
42. $x^2 - 13x = 30$.
43. $3x^2 + 4x = 7$.
44. $3x^2 + 6x = 24$.
45. $4x^2 - 5x = 26$.
46. $5x^2 - 7x = 24$.
47. $2x^2 - 35 = 3x$.
48. $3x^2 - 50 = 5x$.
49. $3x^2 - 24 = 6x$.
50. $2x^2 - 3x = 104$.
51. $2x^2 + 10x = 300$.
52. $3x^2 - 10x = 200$.
53. $4x^2 - 7x + \frac{1}{2} = 0$.
54. $\frac{2}{3}x^2 - \frac{3}{2}x = -\frac{19}{27}$.
55. $9x^2 + 6x - 43 = 0$.
56. $18x^2 - 3x - 66 = 0$.

57. $\frac{1}{2}x^2 - 3x + 1\frac{7}{8} = 0.$

58. $\frac{x^2}{4} - \frac{3x}{2} + 2 = 0.$

59. $2x^2 - 22x = -60.$

60. $3x^2 + 7x - 370 = 0.$

61. $5x^2 - \frac{1}{2}x - \frac{7}{18} = 0.$

62. $\frac{x^2}{3} - \frac{x}{2} + \frac{1}{6} = 0.$

63. $x^2 + 2x + 1 = 6x + 6.$

64. $x^2 - 49 = 10(x - 7).$

65. $2x^2 + 60x = -400.$

66. $a^2 + 7a + 7 = 0.$

67. $z^2 = 3z + 2.$

68. $r = r^2 - 3.$

69. $s^2 = 5s + 6.$

70. $r^2 + 3r = 4.$[1]

71. $2s^2 + 4as - c = 0.$

72. $x^2 + 6ax - 5 = 0.$

73. $x^2 - 10ax = -9a^2.$

74. $cx^2 + 2dx + e = 0.$

75. $2x^2 + 6x - n = 0.$

76. $y^2 + \frac{2}{5}y = \frac{3}{5}.$

77. $x^2 = 5 + \frac{4}{3}x.$

78. $u^2 - \frac{6}{7}u - 1 = 0.$

79. $t^2 + \frac{1}{5}t = \frac{4}{5}.$

80. $r^2 - \frac{5}{2} = \frac{3}{4}r.$

81. $s^2 - \frac{5}{6}s = 2\frac{5}{6}.$

82. $3r^2 - 2r = 40.$

83. $4x^2 - 3x = 3.$

84. $9t^2 + 4t = 6.$

85. $5(x^2 - 25) = x - 5.$

86. $9u^2 + 18u + 8 = 0.$

87. $x^2 + px + q = 0.$

88. $x^4 - 8x^2 + 15 = 0.$

89. $u^4 - 29u^2 + 100 = 0.$

90. $\frac{2}{x^2} = 1 - \frac{8}{3x}.$

91. $2y + \frac{5}{2} = \frac{5}{4y}.$

92. $\frac{1}{6} + \frac{3}{4x} = \frac{5}{4x^2}.$

93. $\frac{5}{5-x} + \frac{8}{8-x} = 3.$

94. $\frac{n-3}{n-2} - \frac{n+4}{n} = \frac{3}{2}.$

95. $3x + \frac{7x+11}{x} = 2 + \frac{33}{x}.$

96. $\frac{24}{x} - \frac{24}{x-2} + 1 = 0.$

97. $x^6 - 35x^3 + 216 = 0.$

98. $x^2 + \frac{100}{x^2} = 29.$

99. $\left(x + \frac{1}{x}\right)^2 - \frac{16}{3}\left(x + \frac{1}{x}\right) + 7 = \frac{1}{3}.$

100. $u + \frac{1}{u} = a + \frac{1}{a}.$

101. $\frac{2x-7}{x^2-4} = \frac{10x-3}{5x(x+2)}.$

HINT: Clear of fractions by multiplying both members of the equation by $5x(x^2 - 4)$, the lowest common denominator.

102. $\frac{2}{2x+1} - \frac{1}{3x+2} + \frac{7}{6x^2+7x+2} = 0.$

103. $\frac{12x-5}{21} - \frac{3x+4}{3(3x+1)} = \frac{4x-5}{7}.$

104. $\frac{x^2+3}{2(x^3-8)} + \frac{1}{6(x-2)} = \frac{2x-1}{3(x^2+2x+4)}.$

105. $\dfrac{x}{x-1}+\dfrac{x-1}{x}=\dfrac{x^2+x-1}{x^2-x}$.

106. $\dfrac{x}{x+2}-\dfrac{x}{x+3}=\dfrac{x^2+2x-2}{x^2+5x+6}$.

107. $\dfrac{1}{x-2}+\dfrac{7x}{24(x+2)}+\dfrac{15}{4-x^2}=0$.

108. $\left(x+\dfrac{1}{x}\right)^2-\frac{16}{3}\left(x+\dfrac{1}{x}\right)+7=\frac{1}{3}$.

HINT: Let $x+\dfrac{1}{x}=y$. Then

$$y^2-\tfrac{16}{3}y+7=\tfrac{1}{3}.$$

Solve this equation for y. Place $x+\dfrac{1}{x}$ equal to each value found for y and solve the resulting equations for x. There are in all four roots of the given equation.

109. $x^6-35x^3+216=0$.
HINT: Let $x^3=y$.

110. $x^2+\dfrac{100}{x^2}=29$.

111. $x+\dfrac{1}{x}=2+\frac{1}{2}$.

112. $\dfrac{24}{x}-\dfrac{24}{x-2}+1=0$.

113. $(x^2+x)^2=12+4(x^2+x)$.

114. $1+\dfrac{x+1}{x-1}=\dfrac{12(x-1)}{x+1}$.

115. $\dfrac{(x-1)(x-2)}{x-3}=\dfrac{(x+1)(x+2)}{x+3}$.

116. Solve $a^2x^2-2hxy+b^2y^2=1$ for y, considering a, b, h, and x as known numbers.

310. The Definitions of Exponents.

(1) n a positive integer: $a^n=aaa\ .\ .\ .$ to n factors.

(2) n and r positive integers: $a^{1/r}=\sqrt[r]{a}$ and $a^{n/r}=(\sqrt[r]{a})^n=\sqrt[r]{a^n}$.

(3) $a^0=1$.

(4) n any number, positive or negative, integral or fractional: $a^{-n}=1/a^n$.

311. The Laws of Exponents. For n and r any numbers, positive or negative, integral or fractional:

(1) $a^n a^r=a^{n+r}$, or law for multiplication and division.

(2) $(a^n)^r = a^{nr}$, or law for involution.

(3) $a^n b^n = (ab)^n$, or distributive law of exponents.

NOTE: The student must distinguish between $-a^n$ and $(-a)^n$. Thus $-8^{\frac{1}{3}} = -2$, and $(-8)^{\frac{1}{3}} = -2$, but $(-3)^2 = 9$ and $-3^2 = -9$.

Exercises 1

Use the definitions of exponents (1), (2), (3), (4) §**307**, and the laws of exponents (1), (2), (3), §**308**, and find the results of the indicated operations in the following exercises.

1. $x^{12}x^{13}$.
2. $a^{2n}a^{3n}$.
3. $x^{2n+1}x^n$.
4. b^2b^{n+5}.
5. $m^{n+1}m^{n-1}$.
6. $a^{n-2}a^{3+n}$.
7. $x^{n-r+1}x^r$.
8. $m^{5a}m^{-2a}$.
9. $x^{13} \div x^8$.
10. $x^8 \div x^{13}$.
11. $a^{3n} \div a^n$.
12. $e^{n+5} \div e^3$.
13. $10^{2r+3} \div 10^r$.
14. $n^{r+6} \div n^{r+3}$.
15. $u^{n+r} \div u^{n-r}$.
16. $x^{n-r+1} \div x^r$.
17. $(a^7)^3$.
18. $(a^4)^6$.
19. $(-ab^2)^3$.
20. $(a^2y^2)^5$.
21. $(b^m)^2$.
22. $(-a^nb^r)^3$.
23. $(a^3b^5)^t$.
24. $(r^ms^n)^p$.
25. $\left(\frac{x^5}{y^6}\right)^4$.
26. $\left(\frac{a^3}{b^7}\right)^n$.
27. $\left(-\frac{x^n}{y^r}\right)^s$.

Exercises 2

Write each of the following sixteen expressions, using fractional exponents in place of radical signs:

1. $\sqrt{a}$.
2. $\sqrt{a^3}$.
3. $\sqrt[3]{a^2}$.
4. $\sqrt[5]{a^3}$.
5. $\sqrt[7]{a^5}$.
6. $(\sqrt{a})^6$.
7. $\sqrt[5]{a^7}$.
8. $(\sqrt[4]{a})^5$.
9. $\sqrt[8]{x^3}$.
10. $(\sqrt[8]{x})^3$.
11. $\sqrt[8]{x^t}$.
12. $(\sqrt[3]{x})^t$.
13. $\sqrt[5]{a-5}$.
14. $(\sqrt[3]{a-b})^t$.
15. $\sqrt{a^2-b^2}$.
16. $\sqrt[3]{(a+b)^t}$.

Find the numerical value of each of the following sixteen expressions:

17. $4^{\frac{1}{2}}$.
18. $27^{\frac{1}{3}}$.
19. $9^{\frac{1}{2}}$.
20. $16^{\frac{1}{4}}$.
21. $625^{\frac{1}{4}}$.
22. $64^{\frac{1}{6}}$.
23. $216^{\frac{1}{3}}$.
24. $16^{\frac{5}{4}}$.
25. $81^{\frac{3}{2}}$.
26. $125^{\frac{5}{3}}$.
27. $32^{\frac{4}{5}}$.
28. $81^{\frac{5}{4}}$.
29. $256^{\frac{3}{4}}$.
30. $64^{\frac{5}{6}}$.
31. $512^{\frac{2}{3}}$.
32. $128^{\frac{4}{7}}$.

Write each of the following expressions *in two ways*, using radical signs instead of fractional exponents:

33. $a^{\frac{1}{3}}$. 37. $n^{\frac{3}{5}}$. 41. $r^{\frac{7}{6}}$. 45. $a^{\frac{n}{3}}$.

34. $1^{\frac{2}{3}}$. 38. $b^{\frac{3}{4}}$. 42. $x^{\frac{p}{q}}$. 46. $b^{\frac{r}{2n}}$.

35. $m^{\frac{4}{7}}$. 39. $e^{\frac{5}{6}}$. 43. $y^{\frac{1}{q}}$. 47. $x^{\frac{n+1}{r}}$.

36. $x^{\frac{2}{7}}$. 40. $h^{\frac{1}{3}}$. 44. $a^{\frac{s}{q}}$. 48. $a^{\frac{r+s}{t}}$.

Exercises 3

Perform the indicated operations in each of the following examples by means of the laws of exponents.

1. $a^{\frac{2}{3}} \times a^{\frac{3}{4}}$.

$$a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{8}{12}+\frac{9}{12}} = a^{\frac{17}{12}}.$$

2. $x^{\frac{2}{5}} \times x^{\frac{3}{2}}$. 4. $x^{\frac{1}{4}} \times x^{\frac{1}{5}}$. 6. $a^{\frac{1}{2n}} \times a^{\frac{4}{3n}}$

3. $x^{\frac{1}{5}} \times x^{\frac{8}{10}}$. 5. $a^{\frac{2}{7}} \times a^{\frac{3}{5}}$. 7. $a^{\frac{2}{n}} \times a^{\frac{r}{2n}}$.

8. $a^{\frac{5}{7}} \div a^{\frac{2}{3}}$.

$$a^{\frac{5}{7}} \div a^{\frac{2}{3}} = a^{\frac{5}{7}-\frac{2}{3}} = a^{\frac{15}{21}-\frac{14}{21}} = a^{\frac{1}{21}}.$$

9. $h^{\frac{2}{3}} \div h^{\frac{1}{3}}$. 11. $8a^5b^{\frac{2}{3}} \div 4a^2b^{\frac{1}{2}}$. 13. $6a^{\frac{2}{3}} \div 3a^{\frac{1}{2}}$.

10. $m^{\frac{3}{5}} \div m^{\frac{3}{10}}$. 12. $9a^{\frac{3}{5}} \div a^{\frac{1}{2}}$. 14. $ab^{\frac{n}{r}} \div a^{\frac{1}{2}}\, b^{\frac{n}{4r}}$.

15. $(a^{\frac{4}{5}})^{\frac{7}{8}}$.

$$(a^{\frac{4}{5}})^{\frac{7}{8}} = a^{\frac{28}{40}} = a^{\frac{7}{10}}.$$

16. $(a^{\frac{2}{3}})^{\frac{6}{10}}$. 18. $(a^{\frac{3}{5}})^{\frac{5}{8}}$. 20. $[(x^n)^{\frac{1}{n}}]^{\frac{n}{r}}$.

17. $(h^{\frac{2}{5}})^{\frac{5}{6}}$. 19. $(a^{\frac{5}{12}})^4$. 21. $(x^{\frac{st}{6rn}})^{\frac{2}{t}}$.

22. $(a^{\frac{2}{3}}x^{\frac{1}{2}}y^{\frac{3}{4}})^{\frac{6}{5}}$.

$$(a^{\frac{2}{3}}x^{\frac{1}{2}}y^{\frac{3}{4}})^{\frac{6}{5}} = (a^{\frac{2}{3}})^{\frac{6}{5}}(x^{\frac{1}{2}})^{\frac{6}{5}}(y^{\frac{3}{4}})^{\frac{6}{5}} = a^{\frac{4}{5}}x^{\frac{3}{5}}y^{\frac{9}{10}}.$$

23. $(a^2b^{\frac{1}{2}})^{\frac{1}{2}}$. 25. $(36a^4x^2y^3)^{\frac{1}{2}}$. 27. $(32x^{\frac{3}{5}}y^{\frac{3}{4}})^{\frac{4}{3}}$.

24. $(ad^{\frac{1}{2}})^{\frac{2}{5}}$. 26. $(a^{\frac{1}{2}}x^{\frac{1}{7}}y^{\frac{1}{8}})^{14}$. 28. $(\frac{1}{8}a^5b^3c)^{\frac{1}{3}}$.

29. $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{3}{4}}}\right)^{\frac{1}{6}}$

$$\left(\frac{a^{\frac{2}{3}}}{b^{\frac{3}{4}}}\right)^{\frac{1}{6}} = \frac{(a^{\frac{2}{3}})^{\frac{1}{6}}}{(b^{\frac{3}{4}})^{\frac{1}{6}}} = \frac{a^{\frac{1}{9}}}{b^{\frac{1}{8}}}.$$

30. $\left(\dfrac{x}{y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$. 32. $\left(\dfrac{a^3}{b^{\frac{2}{3}}}\right)^{\frac{2}{7}}$. 34. $\left(\dfrac{8x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^{\frac{2}{3}}$.

31. $\left(\dfrac{a^2}{b^3}\right)^{\frac{1}{5}}$. 33. $\left(\dfrac{a^5}{b^{\frac{2}{3}}}\right)^{\frac{3}{5}}$. 35. $\left(\dfrac{9a^{\frac{3}{4}}}{4b^{\frac{2}{5}}}\right)^{\frac{3}{2}}$.

36. $(a^{\frac{4}{3}} + a^{\frac{2}{3}} + 1)(a^{\frac{5}{3}} + a - a^{\frac{1}{3}})$.

We arrange the work thus:

$$\begin{array}{l} a^{\frac{4}{3}} + a^{\frac{2}{3}} + 1 \\ a^{\frac{5}{3}} + a \quad - a^{\frac{1}{3}} \\ \hline a^3 + a^{\frac{7}{3}} + a^{\frac{5}{3}} \\ \qquad a^{\frac{7}{3}} + a^{\frac{5}{3}} + a \\ \qquad\qquad - a^{\frac{5}{3}} - a - a^{\frac{1}{3}} \\ \hline a^3 + 2a^{\frac{7}{3}} + a^{\frac{5}{3}} \quad - a^{\frac{1}{3}} \end{array}$$

37. $(x + 2y^{\frac{1}{2}} + 3y^{\frac{1}{3}})(x - 2y^{\frac{1}{2}} + 3y^{\frac{1}{3}})$.

38. $(x^{\frac{3}{4}} + y^{\frac{3}{4}})(x^{\frac{3}{4}} - y^{\frac{3}{4}})$.

39. $(a^{\frac{4}{5}} - 3a^{\frac{3}{5}}b^{\frac{1}{2}} + 4a^{\frac{2}{5}}b - a^{\frac{1}{5}}b^{\frac{3}{2}})(a^{\frac{2}{5}} - 2a^{\frac{1}{5}}b^{\frac{1}{2}})$.

40. $(a^{\frac{3}{n}} - 2a^{\frac{2}{n}} + 3a^{\frac{1}{n}})(2a^{\frac{2}{n}} - a^{\frac{1}{n}})$.

Exercises 4

Find the numerical value of each of the following:

1. 2^{-1}.
2. 4^{-2}.
3. $(-2)^{-3}$.
4. 10^{-5}.
5. 1^{-1}.
6. 2^{-2}.
7. 2^{-4}.
8. $16^{-\frac{3}{4}}$.
9. $81^{-\frac{1}{2}}$.
10. $1024^{-\frac{3}{5}}$.
11. $512^{-\frac{1}{3}}$.
12. $625^{-\frac{3}{4}}$.
13. $\dfrac{1}{2^{-1}}$.
14. $\dfrac{2}{3^{-2}}$.
15. $\dfrac{5}{(-4)^{-2}}$.
16. $\dfrac{1^{-8}}{8^{-1}}$.
17. $\dfrac{5^{-2}}{36^{-\frac{1}{2}}}$.
18. $\dfrac{32^{-\frac{1}{5}}}{2^{-1}}$.
19. $\dfrac{16^{-\frac{3}{4}}}{5^{-2}}$.
20. $\dfrac{7^{-1}}{49^{-\frac{1}{2}}}$.

Write each of the following expressions without using negative exponents:

21. x^{-2}.
22. x^2y^{-2}.
23. $\dfrac{1}{x^{-2}}$.
24. $\dfrac{m^{-3}}{x^{-6}}$.
25. $5a^{-5}$.
26. $3a^{-2}b^{-\frac{1}{2}}$.
27. $\dfrac{2a^{-2}}{3b^2x^{-3}}$.
28. $\dfrac{a^2b^{-4}}{x^{-3}y^{-5}}$.
29. $(x + y)^{-2}$.
30. $(-x)^{-3}$.
31. $\dfrac{x^4}{y^{-5}}$.
32. $\dfrac{3a^{\frac{2}{3}}b^{-1}}{5a^{-\frac{2}{3}}b}$.
33. $2a^3x^{-2}y^{-\frac{1}{2}}$.
34. $(-a^2)^{-3}$.
35. $\dfrac{a^{-\frac{2}{3}}b^{\frac{1}{6}}}{3x^{-\frac{1}{2}}b^{-\frac{1}{6}}}$.
36. $\dfrac{3a^2b^{-2}c^{-4}}{5a^{-1}b^{-3}c^{-5}}$.

Write each of the following expressions *in one line:*

37. $\dfrac{1}{a^2}$.
38. $\dfrac{4}{a^r}$.
39. $\dfrac{ab^2}{c^3d^4}$.
40. $\dfrac{2x^{-1}y^5}{3a^{-2}y^{-3}}$.
41. $\dfrac{3x^3y^{-3}}{4r^{-2}t^5}$.
42. $\dfrac{4x^{-5}y^{-6}}{u^4z^{-7}}$.
43. $\dfrac{x^{-4}y^2}{5a^nb^{-r}}$.
44. $\dfrac{ab^{\frac{1}{2}}}{x^{-\frac{1}{3}}y^{s/t}}$.

45. $\dfrac{16(a+b)^{-1}c^{\frac{1}{2}}}{(a-b)^{-\frac{3}{5}}c^{-\frac{5}{2}}}$.

46. $\dfrac{a}{x^3} + \dfrac{b}{x^2} + \dfrac{c}{x} + \dfrac{d}{x^{-1}}$.

Exercises 5

Perform the indicated operations in each of the following by means of the laws of exponents.

1. $a^8 \times a^{-5}$.
2. $r^{12} \times r^{-10}$.
3. $c^{-8} \div c^{-5}$.
4. $8a^{-4} \times 3a^2$.
5. $u^{-1} \times u^{\frac{2}{3}}$.
6. $x^5 \div x^{-11}$.
7. $m^{-\frac{2}{3}} \times m^{-\frac{1}{3}}$.
8. $6ax^{-6} \times \frac{1}{2}bx^2$.
9. $a^{-3}b^{-n} \div ab^{-n}$.
10. $(-7a^{-3}b^{-2})(-4a^2b^{-1})(a^{-2}b^2x^{-1})$.
11. $(2a^{\frac{1}{2}}b^{-\frac{2}{3}})(a^{-\frac{3}{2}}b^{\frac{2}{3}} - \frac{1}{2}a^{\frac{3}{2}}b^{\frac{1}{2}} + a^{\frac{3}{2}}b^{-\frac{2}{3}})$.
12. $7a^{-1}b^{-2}c^{-3} \div 8a^{-2}b^{-3}c^{-4}$.
13. $56x^5y^{-7}z^4 \div 7x^{-1}y^{-3}z^{-4}$.
14. $18a^{-\frac{1}{2}}b^{\frac{2}{3}}c^{-5} \div 6a^{\frac{3}{2}}b^{\frac{1}{3}}c^{-5}$.
15. $6x^{\frac{1}{3}}y^{-\frac{2}{3}}z^{\frac{1}{6}} \div 2x^{-\frac{2}{3}}y^{\frac{1}{3}}z^{-\frac{1}{2}}$.
16. $(a^{-3})^2$.
17. $(a^{-2})^{-5}$.
18. $(a^5)^{-2}$.
19. $(n^{\frac{1}{3}})^{-3}$.
20. $(r^{-\frac{3}{4}})^{-\frac{2}{3}}$.
21. $(c^{-3})^{\frac{3}{4}}$.
22. $(abc)^{-4}$.
23. $(a^{-4}b^2c^6)^{-3}$.
24. $(x^{\frac{5}{6}}y^{\frac{3}{4}})^{-12}$.
25. $(8x^3y^{-6})^{-\frac{1}{3}}$.
26. $(x^{-\frac{3}{4}}b^{\frac{5}{6}})^{\frac{1}{8}0}$.
27. $(a^{-5}b^{-10})^{-\frac{2}{5}}$.
28. $(-1x^3)^{-4}$.
29. $(-a^{-4})^{-3}$.
30. $(-a^6)^{-\frac{4}{3}}$.
31. $(-8^{-3}b^{\frac{3}{4}}c^{-\frac{1}{3}})^{-\frac{1}{3}}$.
32. $(-1x^3y^{-9})^{\frac{1}{3}}$.
33. $\left(\frac{a}{b}\right)^{-5}$.
34. $\left(\frac{a^{-5}}{a^{-4}}\right)^{-7}$.
35. $\left(\frac{b^{-4}}{b^{-6}}\right)^{-2}$.
36. $\left(\frac{x^3}{y^9}\right)^{-\frac{1}{2}}$.
37. $\left(\frac{a^{-2}b^3}{x^{-1}y^{-4}}\right)^2$.
38. $\left(\frac{a^{-3}b}{x^3y^{-2}}\right)^{-3}$.
39. $\left(\frac{a^2b^{-2}}{xy^3}\right)^3$.
40. $\left(\frac{a^5b^{-3}}{3x^{-1}y^2}\right)^{-1}$.
41. $\left(\frac{a^6b^{-3}}{x^{-9}y^{-\frac{3}{4}}}\right)^{-\frac{1}{3}}$.
42. $(a^2x^{-1}+3a^3x^{-2})(4a^{-1} - 5x^{-1} + 6ax^{-2})$

$$\begin{array}{r} 4a^{-1} - 5x^{-1} + 6ax^{-2} \\ a^2x^{-1} + 3a^3x^{-2} \\ \hline 4ax^{-1} - 5a^2x^{-2} + 6a^3x^{-3} \\ 12a^2x^{-2} - 15a^3x^{-3} + 18a^4x^{-4} \\ \hline 4ax^{-1} + 7a^2x^{-2} - 9a^3x^{-3} + 18a^4x^{-4} \end{array}$$

43. $(2x^{-\frac{1}{5}} - 3x + 4x^{\frac{1}{5}})(x^{-\frac{3}{5}} - 2x^{-\frac{2}{5}} + 3x^{-\frac{1}{5}})$.
44. $(x^{-\frac{4}{3}} - 2x^{-\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}})(x^{-\frac{2}{3}} - y^{\frac{2}{3}})$.
45. $(3x^{\frac{2}{3}} - \frac{3}{2}x^{\frac{1}{3}} + 4) \times 2x^{-\frac{1}{3}}$.
46. $(x^{-\frac{2}{3}} + x^{-\frac{1}{3}} + 1)(x^{-\frac{1}{3}} - 1)$.
47. $(x^{-\frac{1}{2}} + y^{-2})(x^{-\frac{1}{2}} - y^{-2})$.
48. $(x^{\frac{1}{4}}y + y^{\frac{3}{5}})(x^{\frac{1}{4}} - y^{-\frac{1}{5}})$.

49. $(2a^{\frac{3}{2}} - 3ax^{\frac{1}{2}})(3a^{-\frac{1}{2}} + 2x^{-\frac{1}{2}})(4a^{\frac{1}{2}}x^{\frac{1}{2}} + 9a^{-\frac{1}{2}}x^{\frac{3}{2}})$.

50. $(x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y - y^{\frac{3}{2}}) \div (x^{-\frac{1}{2}} - y^{\frac{1}{2}})$.

$$x^{-\frac{1}{2}} - y^{\frac{1}{2}})x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y - y^{\frac{3}{2}}(x^{-1} + y$$

$$\underline{x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}}}$$

$$x^{-\frac{1}{2}}y - y^{\frac{3}{2}}$$

$$\underline{x^{-\frac{1}{2}}y - y^{\frac{3}{2}}}$$

51. $(x^{-3} + 2x^{-2} - 3x^{-1}) \div (x^{-2} + 3x^{-1})$.

312. Reduction of Surds or Radicals.

1. *If any factor of the number under the radical sign is an exact power of the indicated root, the root of that factor may be extracted and written as the coefficient of the surd, while the other factors are left under the radical sign.*

(1) Thus,
$$\begin{aligned}\sqrt{8} &= \sqrt{4 \times 2}\\ &= \sqrt{4}\sqrt{2}\\ &= 2\sqrt{2}\end{aligned}$$

(2) Also,
$$\begin{aligned}\sqrt[3]{81} &= \sqrt[3]{27 \times 3}\\ &= \sqrt[3]{27}\sqrt[3]{3}\\ &= 3\sqrt[3]{3}\end{aligned}$$

(3) Also,
$$\begin{aligned}\sqrt[3]{16ax^4} &= \sqrt[3]{8x^3 \times 2ax}\\ &= \sqrt[3]{8x^3}\sqrt[3]{2ax}\\ &= 2x\sqrt[3]{2ax}\end{aligned}$$

2. *The expression under the radical sign of any surd can always be made integral.*

(1) Thus
$$\begin{aligned}\sqrt[3]{\frac{2}{3}} &= \sqrt[3]{\frac{2}{3} \times \frac{9}{9}} = \sqrt[3]{\frac{18}{27}}\\ &= \sqrt[3]{\frac{1}{27} \times 18}\\ &= \frac{1}{3}\sqrt[3]{18}\end{aligned}$$

(2) Also
$$\begin{aligned}\sqrt[4]{\frac{7}{8}} &= \sqrt[4]{\frac{7}{8} \times \frac{2}{2}} = \sqrt[4]{\frac{1}{16} \times 14}\\ &= \frac{1}{2}\sqrt[4]{14}\end{aligned}$$

3. *We may change the index of some surds in the following manner:*

(1) **Thus,** $\sqrt[4]{4} = \sqrt{\sqrt{4}}$

$= \sqrt{2}$

(2) **Also,** $\sqrt[6]{1000} = \sqrt{\sqrt[3]{1000}}$

$= \sqrt{10}$

(3) **Also,** $\sqrt[6]{256c^2a^8} = \sqrt[3]{\sqrt{256c^2a^8}}$

$= \sqrt[3]{16ca^4}.$

A surd is in its **simplest form** when (1) no factor of the expression under the radical sign is a perfect power of the required root, (2) the expression under the radical sign is integral, (3) the index of the surd is the lowest possible.

Methods of making the different reductions required by this definition have already been explained. We give a few examples.

(1) Simplify $\sqrt[9]{\frac{a^3}{8b^6}}$.

$$\sqrt[9]{\frac{a^3}{8b^6}} = \sqrt[3]{\frac{a}{2b^2}} = \frac{1}{2b}\sqrt[3]{4ab}.$$

(2) Simplify $\sqrt[4]{\frac{400}{9}}$.

$$\sqrt[4]{\frac{400}{9}} = \sqrt{\frac{20}{3}} = \frac{1}{3}\sqrt{60} = \frac{2}{3}\sqrt{15}.$$

(3) Simplify $\frac{5}{2}\sqrt[6]{\frac{512}{125}}$.

$$\frac{5}{2}\sqrt[6]{\frac{512}{125}} = \frac{5}{2}\sqrt{\frac{8}{5}} = 5\sqrt{\frac{2}{5}} = \sqrt{10}.$$

(4) Simplify $3\sqrt{2} + 2\sqrt{\frac{1}{2}} + \sqrt{8}$.

$$3\sqrt{2} + 2\sqrt{\frac{1}{2}} + \sqrt{8} = 3\sqrt{2} + \sqrt{2} + 2\sqrt{2}$$
$$= 6\sqrt{2}.$$

313. Rationalizing the Denominator of a Fraction.

ILLUSTRATION: Rationalize the denominator of

$$\frac{\sqrt{3}+2}{\sqrt{3}-\sqrt{2}} = \frac{(\sqrt{3}+2)(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{3+2\sqrt{3}+\sqrt{6}+2\sqrt{2}}{1}.$$

In any piece of work it is usually expected that all the surds will finally be left in their simplest form.

Exercises

Reduce each of the following surds to its simplest form:

1. $\sqrt{\frac{9}{5}}$.
2. $\sqrt[3]{\frac{16}{3}}$.
3. $\sqrt[4]{\frac{25}{4}}$.
4. $\sqrt[6]{\frac{64}{27}}$.
5. $\sqrt[6]{\frac{64}{81}}$.
6. $\sqrt[3]{\frac{5}{12}}$.
7. $\sqrt[4]{\frac{a^2}{b^4x^6}}$.
8. $\sqrt{1-\frac{1}{x^2}}$.
9. Simplify $\sqrt{12}+\frac{1}{10}\sqrt{75}+6\sqrt{\frac{1}{12}}$.
10. Simplify $1+\sqrt{8}+\sqrt{2}-\sqrt{27}-\sqrt{12}+\sqrt{75}$.
11. Simplify $\sqrt[3]{21+7\sqrt{2}} \times \sqrt[3]{21-7\sqrt{2}}$.
12. Find the value of x^2-6x+7 if $x=3-\sqrt{3}$.
13. Find the value when $x=\sqrt{3}$ of the expression
$$\frac{2x-1}{(x-1)^2}-\frac{2x+1}{(x+1)^2}.$$
14. Find the value of
$$(35\sqrt{10}+77\sqrt{2}+63\sqrt{3})(\sqrt{10}+\sqrt{2}+\sqrt{3}).$$

Solve and check each of the following equations:

15. $\sqrt{x+4}=4$.
16. $\sqrt{2x+6}=4$.
17. $\sqrt{10x+16}=5$.
18. $\sqrt{2x+7}=\sqrt{5x-2}$.
19. $14+\sqrt[3]{4x-40}=10$.
20. $\sqrt{16x+9}=4\sqrt{4x}-3$.
21. $\sqrt{\frac{15}{4}+x}=\frac{3}{2}+\sqrt{x}$.
22. $\sqrt{x}-\sqrt{x-5}=\sqrt{5}$.
23. $\sqrt{x-7}=\sqrt{x-14}+1$.
24. $\sqrt{x-7}=\sqrt{x+1}-2$.
25. $x=7-\sqrt{x^2-7}$.
26. $\sqrt{x+20}-\sqrt{x-1}-3=0$.
27. $\sqrt{x+3}+\sqrt{3x-2}=7$.
28. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}$.
29. $\frac{20x}{\sqrt{10x-9}}-\sqrt{10x-9}=\frac{18}{\sqrt{10x-9}}+9$.
30. $\frac{x-1}{\sqrt{x}-1}=\frac{\sqrt{x}+1}{x-3}$.
31. $\frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}-\sqrt{x-3}}=\frac{3}{x-3}$.

Exercises

Rationalize the denominator of each of the following:

1. $\dfrac{6}{3+\sqrt{5}}$.
2. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.
3. $\dfrac{2\sqrt{2}+3}{3\sqrt{2}+2}$.
4. $\dfrac{5\sqrt{2}+6}{3\sqrt{2}-6}$.
5. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.
6. $\dfrac{\sqrt{x-2}+1}{\sqrt{x-2}+2}$.
7. $\dfrac{\sqrt{a-b}+\sqrt{a}}{\sqrt{a-b}-\sqrt{a}}$.
8. $\dfrac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}}$.
9. $\dfrac{1}{x^2-\sqrt{1+x^2}}$.
10. $\dfrac{x^2}{1+\sqrt{1-x^2}}$.

Miscellaneous Exercises

Solve the following simultaneous equations:

1. $3x+2y=8,$
   $2x-5y=-1.$
2. $8x-5y=49,$
   $7x+15y=101.$
3. $5x-3y=47,$
   $13x+5y=135.$
4. $\frac{1}{3}x+\frac{1}{4}y=6,$
   $3x-4y=4.$
5. $ax+by=5,$
   $ax-by=3.$
6. $\dfrac{x+1}{y}=a,$
   $\dfrac{y+1}{x}=b.$

Solve by formula for quadratic equations:

7. $9x^2+17x=310.$
8. $4x^2-6dx=4d^2.$
9. $1.3x^2-2.5x+5.9=0.$
10. $0.3+0.5x=0.2x^2.$

11. A stone is dropped into a well. The sound of its impact upon the water is heard at the top of the well 4.2 seconds later. If the velocity of sound is 1130 feet per second and if a body falls $\frac{1}{2}gt^2$ feet in t seconds, how deep is the well? Take $g=32$ feet per second per second.

Solve the following simultaneous equations:

12. $x+y=5,$
    $xy=6.$
13. $x-y=5,$
    $xy=176.$
14. $x^2-3xy+y^2=-11,$
    $x^2+y^2=34.$
15. $xy-5x=1,$
    $7x-y=1.$
16. $x^2+y^2=40,$
    $3x-y=0.$
17. $2x+3y=7,$
    $x^2+2y^2-y=5.$

18. $3x^2 - 4xy + 3y^2 = 2,$ (1)
$2x^2 - 2xy + 5y^2 = 5.$ (2)

HINT: Eliminate the 2 and 5 on right sides of the equations by subtraction. This gives

$$11x^2 - 16xy + 5y^2 = 0 \quad (3)$$
or
$$(x - y)(11x - 5y) = 0 \quad (4)$$

which gives $x = y$ and $x = \frac{5}{11}y$. Substitute each of these in (1) or (2) and solve the resulting quadratic.

19. $x^2 - 9y^2 = 7,$
$xy + 4y^2 = 8.$

20. $2x^2 - 3xy + y^2 = 3,$
$x^2 + 2xy - 3y^2 = 5.$

Simplify the following:

21. $2\sqrt[3]{\frac{3}{5}} + \sqrt{60} - \sqrt{15} + \sqrt[3]{\frac{3}{5}}$. **22.** $(9 + 2\sqrt{10})(9 - 2\sqrt{10})$.
23. $(\frac{1}{3} - 6a^{-2} + 27a^{-4}) \times (2a^{\frac{1}{2}} + 3a^{\frac{3}{2}})$.

Factor:

24. $x^4 - (a + 4)x^2 + 4a.$
25. $9(x - 2y)^2 - 4(x - 3y)^2.$
26. $10a^4 - 33a^3 - 7a^2.$
27. $128 - 2a^2.$
28. $a^3 - ab^2.$
29. $a^6 - 729.$

Simplify:

30. $\dfrac{a^2 + ab}{a^2 - ab} \div \dfrac{a^3 + a^2b}{ab^2 + b^3}.$

31. $\dfrac{a^2 - b^2}{(a + b)^2} \div \dfrac{(a - b)^2}{2a}.$

32. $\dfrac{x^2 + 14x - 15}{x^2 + 4x - 5} \div \dfrac{x^2 + 12x - 45}{x^2 + 6x - 27}.$

33. Does $\sqrt{a^2 + b^2} = a + b$?
34. Does $(1 + x)(1 + y) = 1 + xy$?

Remove brackets:

35. $(x^3 - x^2 + x + 1) - (x^3 + x^2 - x + 1).$
36. $(-3a^3 + 5b^3 - c^3) - (a^3 + b^3 - c^3 - 3abc).$
37. $3x - 2(2x^2 - 3y) - 3(x + 2x^2 - y).$
38. $x^2 - (x + 5) - (x - 1).$
39. $5 - (y + [3 - 5y]).$
40. $a - (b - [c - d]).$

Put in parentheses, two by two, beginning with first term.

41. $a - b + c - d - e - f + g - h.$
42. $a^3 - b^3 - a^2 - b^2 + a - b.$
43. $-n + 1 + n^2 - 1 - n^3 + 1.$
44. $x - a - x^2 + ax + 2ax - a^2.$

Divide (use long division):

45. $\dfrac{x^2 - 3x + 2}{x - 1}$.

46. $\dfrac{6a^2 - 16a + 8}{2a - 4}$.

47. $\dfrac{16a^2 - 25b^2}{4a - 5b}$.

48. $\dfrac{2a^4 + 8a - a^3 + 15}{2a^2 - 3a + 5}$.

49. $\dfrac{x^2 - y^2 - 2yz - z^2}{x - y - z}$.

50. $\dfrac{2x^3 - 10 - 6x^2 + x^4 + 11x}{2 + x^2 - x}$.

51. $\dfrac{243n^5 + 1}{3n + 1}$.

52. $\dfrac{3n^4 - 11n^3 - 25n^2 + 2 - 13n}{3n^2 + 4n + 1}$.

53. $(x^2 + 5x + 10) \div (x + 2)$.

Factor:

54. $25(a - b)^2 - (c + d)^2$.

55. $a^2 + 2ab + b^2 - m^2 - 2mn - n^2$.

56. $6x^3 - 9x^2 + 3x$.

57. $x^2z^2 - 20xz - 69$.

58. $a^2 - 1 - x(a - 1)$.

59. $6x^2 + 17x + 7$.

60. $27a^3 + 64b^3$.

61. $6a^3 + 4a^2 - 9a - 6$.

62. $x^4 + 16b^4$.

63. $x^4 - 5x^2y^2 + 4y^4$.

64. $m^4 + m^2n^2 + n^4$.

65. $1 - x^2 - 2xy - y^2$.

66. $2(a - 2)^2 - 5(a - 2) - 3$.

67. $18a^2 + 9a - 2$.

68. $(c + d)^3 - 8$.

69. $(2m - 5n)^2 - (m - 2n)^2$

70. $a^2 + 2ab + b^2 - c^2$.

71. $a^3 + a^2 + a$.

72. $a^2c - a^2c^2 + a^3c^3 + ac^4$.

73. $9a^2 - (b - c)^2$.

74. $x^8 - y^8$.

75. $7x^7 - 7x$.

76. $x^4 + 4y^4 + 4x^2y^2$.

77. $x^8 - y^4$.

78. $4a^2 - (x + y)^2$.

79. $24a^2b^2 - 36b^4 - 30ab^3$.

80. $a^4 + 4b^4$.

81. $x^4 - 7x^2y^2 + y^4$.

82. $6c^2 - c(d - 1) - (d - 1)^2$.

83. $55x^5 - x^4 - 2x^3$.

84. $x^5 - x^4 - 42x^3$.

85. $2a^3 - a^5 + 10a^7 - a^6 + a^2$.

86. $27 - a^5 + 10a^7 - a^6 + a^2$.

Add:

87. $\frac{5x}{3} + \frac{3x}{4} + \frac{x}{2}$.

88. $\frac{4x+1}{4} + \frac{2x-1}{3}$.

89. $\frac{2}{x-1} + \frac{1}{x+1} - \frac{1}{x^2-1}$.

90. $\frac{5a+1}{a+3} - \frac{2a+1}{a-3}$.

91. $\frac{4a-b}{a+2b} + \frac{18ab}{a^2-4b^2} + \frac{4a+b}{a-2b}$.

92. $\frac{2}{3} - \frac{1}{2} + \frac{1}{4}$.

93. $\frac{1}{x(x+1)} + \frac{1}{x(x-1)}$.

94. $\frac{a+1}{a-1} + \frac{a-1}{a+1}$.

95. $\frac{3}{m} - \frac{5}{m-1} - \frac{2m-3}{m^2-1}$.

96. $\frac{x^2+x}{x^3-1} - \frac{1}{x-1}$.

97. $\frac{x^3}{x^2+x+1} + \frac{x}{x-1} + \frac{1}{x^3-1}$.

98. $\frac{a}{a+2} + 1 - \frac{a}{a-2} - a$.

99. $\frac{x^2-1}{x^3+x} + \frac{1}{x} + \frac{2x}{x^2+1}$.

100. $\left(2a - \frac{a+3}{2}\right) + \left(a - \frac{a-4}{3}\right) - 2a$.

101. $\frac{a+1}{(a-4)(2-c)} + \frac{a+4}{(a-1)(2-c)}$.

102. $\frac{mn}{m+n} \cdot \left(\frac{m}{n} - \frac{n}{m}\right)$.

103. $\left(\frac{c+d}{c-d} - \frac{c-d}{c+d}\right) \div \left(\frac{6}{c} + \frac{3}{c-d} - \frac{3}{c+d}\right)$.

104. $\left(a + \frac{m^2}{a}\right)\left(\frac{a^2}{m^2} - 1\right) \div \left(1 - \frac{m^4}{a^4}\right)$.

Multiply:

105. $\frac{a^2-144}{a^2-9} \cdot \frac{a+3}{a+12}$.

106. $\frac{m^2-n^2}{m^2} \cdot \frac{m^3}{(m-n)^2}$.

107. $\frac{(c+d)^3}{c^3+d^3} \cdot \frac{c^2-cd+d^2}{c^2+2cd+d^2}$.

108. $\frac{m^2-4}{(m+2)(m-1)} \cdot \frac{m^2-2m+1}{(m+3)(m-2)}$.

Reduce to simplest form:

109. $\sqrt{8}$.

110. $\sqrt{20}$.

111. $\sqrt{48}$.

112. $\sqrt{72}$.

113. $\sqrt[3]{16}$.

114. $-\sqrt[3]{54}$.

115. $-\sqrt[3]{108}$.

116. $\sqrt[3]{-81}$.

117. $\sqrt[4]{162}$.

118. $-\sqrt[5]{96}$.

119. $\sqrt{27a^5}$.

120. $\sqrt[3]{54a^4b^5}$.

121. $-\sqrt[3]{128m^4n^3}$.

122. $-\sqrt[3]{320a^3b^5c^7}$.

123. $3\sqrt{27a}$.

124. $-\frac{1}{3}\sqrt{20bc^3d}$. **125.** $-\frac{1}{5}\sqrt{125c^{17}}$. **126.** $5m\sqrt[3]{(a-1)^4}$.

127. $\sqrt{a^3+2a^2x+ax^2}$. **128.** $\sqrt{36(a^2-x^2)(a+x)}$.

129. $\sqrt{\frac{1}{2}}$. **130.** $\sqrt{\frac{1}{3}}$. **131.** $\sqrt{\frac{2}{3}}$.

132. $\sqrt{\frac{3}{5}}$. **133.** $-3\sqrt[3]{\frac{2}{4}\frac{7}{}}$. **134.** $\frac{1}{2}\sqrt{\frac{30}{7}}$.

135. $15\sqrt{\frac{7a^3}{90}}$. **136.** $8\sqrt{\frac{5a^5}{12a^2}}$. **137.** $4\sqrt{288}$.

138. $3\sqrt{98}$. **139.** $\sqrt{3}+4\sqrt{3}$. **140.** $\sqrt{3}+\sqrt{12}$.

141. $\sqrt{10}+\sqrt{40}+\sqrt{90}$.

Multiply:

142. $(\sqrt{20}+\sqrt{80}+\sqrt{45})(\sqrt{5})$.

143. $(\sqrt{8}-2\sqrt{12}+\sqrt{20})\sqrt{6}$.

144. $(2\sqrt{3}-2)(2\sqrt{3}+2)$.

145. $(2\sqrt{5}+3\sqrt{2})(3\sqrt{5}-4\sqrt{2})$

Divide:

146. $2\sqrt{32}$ by $3\sqrt{120}$. **148.** $\sqrt[3]{81}$ by $\sqrt{3}$.

147. $\sqrt[3]{12}$ by $4\sqrt{2}$. **149.** $(12\sqrt{5}-8\sqrt{15}+3\sqrt{30})$ by $6\sqrt{10}$.

Perform indicated operations:

150. $(\sqrt{2}+\sqrt{3})^2$. **152.** $(\sqrt{a+1}-2)^2$.

151. $(\sqrt{3}-2\sqrt{2})^2$. **153.** $(\sqrt{m+2}+\sqrt{m})(\sqrt{m+2}-2\sqrt{m})$.

Rationalize the denominators of:

154. $\frac{4}{\sqrt{2}}$. **155.** $\frac{6}{\sqrt{3}}$. **156.** $\frac{2}{2+\sqrt{2}}$.

157. $\frac{3}{3-\sqrt{3}}$. **158.** $\frac{5}{\sqrt{3}-2}$. **159.** $\frac{2}{\sqrt{2}+\sqrt{3}}$.

160. $\frac{2+\sqrt{3}}{2-\sqrt{3}}$. **161.** $\frac{3\sqrt{2}+2}{2\sqrt{2}-1}$. **162.** $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

163. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. **164.** $\frac{3\sqrt{5}-2\sqrt{6}}{2\sqrt{5}-2\sqrt{6}}$. **165.** $\frac{a\sqrt{c}-b\sqrt{x}}{a\sqrt{c}+b\sqrt{x}}$.

Simplify:

166. $\sqrt{15-6\sqrt{6}}$. **167.** $\frac{\sqrt{8}}{(\sqrt{6}-\sqrt{2})(\sqrt{3}+1)}$.

168. $\frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}$.

Solve the following equations for x. Check.

169. $3x+4=2x+5$.

170. $4x+3=2x+9$.

171. $4 + 5(x + 2) - 9x = (x + 2)^2 - x^2$.

172. $(x + 2)(x - 5) = (x + 4)(x - 1)$.

173. $(x - 1)(x + 3) - 2(x + 1)(x - 5) + x^2 = 0$.

174. $2[x + x)(x - 3) + 1] = (2x + 5)(x - 1)$.

175. $ax + a = 4a$.

176. $(x - m)(x - n) = (x - m - n)^2$.

177. $\dfrac{2x}{3} - \dfrac{x}{5} = \dfrac{x}{2} - \dfrac{1}{6}$.

178. $\frac{1}{2}(x + 2) = \frac{1}{3}(x - 3)$.

179. $(x + \frac{1}{2})(x - \frac{1}{3}) = x^2$.

180. $\dfrac{2(x + 3)}{5} - \dfrac{3(x + 1)}{2} = 0$.

181. $\dfrac{2x - 1}{3x + 4} - \dfrac{4x - 1}{6x - 1} - 1 = \dfrac{-18x^2}{21x - 4 + 18x^2}$.

182. $\dfrac{2}{x + 3} - \dfrac{3}{2(x + 3)} + \dfrac{4}{3(x + 3)} = \dfrac{1}{4}$.

183. $\dfrac{3x - 2}{6} + \dfrac{2x - 1}{2x + 3} = \dfrac{x}{2}$.

184. $\sqrt{x} - 5 = 0$.

185. $x + \sqrt{x + 6} = 14$.

186. $\sqrt{5x + 1} = 2 - \sqrt{x}$.

187. $\sqrt{x + 5} + \sqrt{x - 4} = 9$.

188. $4x = \dfrac{x}{\sqrt{1 - x^2}}$.

189. $5x + y = 11$.
$3x + 2y = 1$.

190. $5x - y = 16$.
$x = y$.

191. $y - x - 1 = 3$.
$x - 5 = -y$.

192. $\dfrac{x}{2} + \dfrac{y}{3} = 5$.

$\dfrac{x}{6} + \dfrac{y}{3} = 3$.

193. $x + 3y = 5$.
$3x + 4y = 0$.

194. $x - y = 0$.
$4x - 5y + 2 = 0$.

195. $3x = -2y$.
$x = 35 + 11y$.

196. $\dfrac{x}{2} + \dfrac{y}{3} = -2$.

$\dfrac{x}{3} - \dfrac{y}{2} = \dfrac{5}{6}$.

197. $ax + by = 1$.
$ax - by = 3$.

198. $x + y = m$.
$2x - 3y = n$.

199. $\dfrac{x}{2} = \dfrac{y}{3}$.

$\dfrac{x}{6} + \dfrac{y}{3} = 5$.

200. $3x + y - 2z = 1$.
$2x - 3y + z = -1$.
$4x - 2y + 3z = 14$.

Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	9	13	17	21	25	30	34	38
											4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	12	15	19	23	27	31	35
											4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	11	14	18	21	25	28	32
											3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	7	10	13	16	20	23	26	30
											3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	28
											3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	9	11	14	17	20	23	26
											3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	14	16	19	22	24
											3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3	5	8	10	13	15	18	20	23
											2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
											2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
											2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

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Logarithms of Trigonometric Functions

°	′	log sin	d	log tan	dc	log cot	log cos			′	S	T
0	0						0.0000	0	90	0		
	10	7.4637	3011	7.4637	3011	2.5363	0.0000	50		10	6.4637	6.4637
	20	7.7648	1760	7.7648	1761	2.2352	0.0000	40		20	6.4637	6.4637
	30	7.9408	1250	7.9409	1249	2.0591	0.0000	30		30	6.4637	6.4637
	40	8.0658	969	8.0658	969	1.9342	0.0000	20		40	6.4637	6.4637
	50	8.1627	792	8.1627	792	1.8373	0.0000	10		50	6.4637	6.4638
1	0	8.2419	669	8.2419	670	1.7581	9.9999	0	89	60	6.4637	6.4638
	10	8.3088	580	8.3089	580	1.6911	9.9999	50		70	6.4637	6.4638
	20	8.3668	511	8.3669	512	1.6331	9.9999	40		80	6.4637	6.4638
	30	8.4179	458	8.4181	457	1.5819	9.9999	30		90	6.4637	6.4638
	40	8.4637	413	8.4638	415	1.5362	9.9998	20		100	6.4637	6.4638
	50	8.5050	378	8.5053	378	1.4947	9.9998	10		110	6.4637	6.4639
2	0	8.5428	348	8.5431	348	1.4569	9.9997	0	88	120	6.4636	6.4639
	10	8.5776	321	8.5779	322	1.4221	9.9997	50		130	6.4636	6.4639
	20	8.6097	300	8.6101	300	1.3899	9.9996	40		140	6.4636	6.4640
	30	8.6397	280	8.6401	281	1.3599	9.9996	30		150	6.4636	6.4640
	40	8.6677	263	8.6682	263	1.3318	9.9995	20		160	6.4636	6.4640
	50	8.6940	248	8.6945	249	1.3055	9.9995	10		170	6.4635	6.4641
3	0	8.7188	235	8.7194	235	1.2806	9.9994	0	87	180	6.4635	6.4641
	10	8.7423	222	8.7429	223	1.2571	9.9993	50		190	6.4635	6.4642
	20	8.7645	212	8.7652	213	1.2348	9.9993	40		200	6.4635	6.4642
	30	8.7857	202	8.7865	202	1.2135	9.9992	30		210	6.4635	6.4643
	40	8.8059	192	8.8067	194	1.1933	9.9991	20		220	6.4634	6.4643
	50	8.8251	185	8.8261	185	1.1739	9.9990	10		230	6.4634	6.4644
4	0	8.8436	177	8.8446	178	1.1554	9.9989	0	86	240	6.4634	6.4644
	10	8.8613	170	8.8624	171	1.1376	9.9989	50		250	6.4633	6.4645
	20	8.8783	163	8.8795	165	1.1205	9.9988	40		260	6.4633	6.4646
	30	8.8946	158	8.8960	158	1.1040	9.9987	30		270	6.4633	6.4646
	40	8.9104	152	8.9118	154	1.0882	9.9986	20		280	6.4632	6.4647
	50	8.9256	147	8.9272	148	1.0728	9.9985	10		290	6.4632	6.4648
5	0	8.9403		8.9420		1.0580	9.9983	0	85	300	6.4631	6.4649
		log cos	d	log cot	dc	log tan	log sin	′	°			

	143	142	138	137	135	134	130	129	127	125	123	122	119	117	115	114
1	14.3	14.2	13.8	13.7	13.5	13.4	13.0	12.9	12.7	12.5	12.3	12.2	11.9	11.7	11.5	11.4
2	28.6	28.4	27.6	27.4	27.0	26.8	26.0	25.8	25.4	25.0	24.6	24.4	23.8	23.4	23.0	22.8
3	42.9	42.6	41.4	41.1	40.5	40.2	39.0	38.7	38.1	37.5	36.9	36.6	35.7	35.1	34.5	34.2
4	57.2	56.8	55.2	54.8	54.0	53.6	52.0	51.6	50.8	50.0	49.2	48.8	47.6	46.8	46.0	45.6
5	71.5	71.0	69.0	68.5	67.5	67.0	65.0	64.5	63.5	62.5	61.5	61.0	59.5	58.5	57.5	57.0
6	85.8	85.2	82.8	82.2	81.0	80.4	78.0	77.4	76.2	75.0	73.8	73.2	71.4	70.2	69.0	68.4
7	100.1	99.4	96.6	95.9	94.5	93.8	91.0	90.3	88.9	87.5	86.1	85.4	83.3	81.9	80.5	79.8
8	114.4	113.6	110.4	109.6	108.0	107.2	104.0	103.2	101.6	100.0	98.4	97.6	95.2	93.6	92.0	91.2
9	128.7	127.8	124.2	123.3	121.5	120.6	117.0	116.1	114.3	112.5	110.7	109.8	107.1	105.3	103.5	102.6

Formulas for using Table directly

For $x < 45°$:
- $\log \sin x = \log x' + S$
- $\log \tan x = \log x' + T$
- $\log \cot x = \operatorname{colog} x' + \operatorname{co} T$

For $x > 45°$:
- $\log \cos x = \log (90 - x)' + S$
- $\log \cot x = \log (90 - x)' + T$
- $\log \tan x = \operatorname{colog} (90 - x)' + \operatorname{co} T$

Logarithms of Trigonometric Functions

° ′	log sin	d	log tan	dc	log cot	log cos		pp
5 0	8.9403	142	8.9420	143	1.0580	9.9983	0 85	113 111 109
10	8.9545	137	8.9563	138	1.0437	9.9982	50	1 11.3 11.1 10.9
								2 22.6 22.2 21.8
20	8.9682	134	8.9701	135	1.0299	9.9981	40	3 33.9 33.3 32.7
30	8.9816	129	8.9836	130	1.0164	9.9980	30	
40	8.9945	125	8.9966	127	1.0034	9.9979	20	4 45.2 44.4 43.6
								5 56.5 55.5 54.5
50	9.0070	122	9.0093	123	0.9907	9.9977	10	6 67.8 66.6 65.4
6 0	9.0192	119	9.0216	120	0.9784	9.9976	0 84	7 79.1 77.7 76.3
10	9.0311	115	9.0336	117	0.9664	9.9975	50	8 90.4 88.8 87.2
								9 101.7 99.9 98.1
20	9.0426	113	9.0453	114	0.9547	9.9973	40	108 107 105
30	9.0539	109	9.0567	111	0.9433	9.9972	30	1 10.8 10.7 10.5
40	9.0648	107	9.0678	108	0.9322	9.9971	20	2 21.6 21.4 21.0
								3 32.4 32.1 31.5
50	9.0755	104	9.0786	105	0.9214	9.9969	10	4 43.2 42.8 42.0
7 0	9.0859	102	9.0891	104	0.9109	9.9968	0 83	5 54.0 53.5 52.5
10	9.0961	99	9.0995	101	0.9005	9.9966	50	6 64.8 64.2 63.0
								7 75.6 74.9 73.5
20	9.1060	97	9.1096	98	0.8904	9.9964	40	8 86.4 85.6 84.0
30	9.1157	95	9.1194	97	0.8806	9.9963	30	9 97.2 96.3 94.5
40	9.1252	93	9.1291	94	0.8709	9.9961	20	
								104 102 101
50	9.1345	91	9.1385	93	0.8615	9.9959	10	1 10.4 10.2 10.1
8 0	9.1436	89	9.1478	91	0.8522	9.9958	0 82	2 20.8 20.4 20.2
10	9.1525	87	9.1569	89	0.8431	9.9956	50	3 31.2 30.6 30.3
								4 41.6 40.8 40.4
20	9.1612	85	9.1658	87	0.8342	9.9954	40	5 52.0 51.0 50.5
30	9.1697	84	9.1745	86	0.8255	9.9952	30	6 62.4 61.2 60.6
40	9.1781	82	9.1831	84	0.8169	9.9950	20	
								7 72.8 71.4 70.7
50	9.1863	80	9.1915	82	0.8085	9.9948	10	8 83.2 81.6 80.8
9 0	9.1943	79	9.1997	81	0.8003	9.9946	0 81	9 93.6 91.8 90.9
10	9.2022	78	9.2078	80	0.7922	9.9944	50	
								99 98 97 95
20	9.2100	76	9.2158	78	0.7842	9.9942	40	1 9.9 9.8 9.7 9.5
30	9.2176	75	9.2236	77	0.7764	9.9940	30	2 19.8 19.6 19.4 19.0
40	9.2251	73	9.2313	76	0.7687	9.9938	20	3 29.7 29.4 29.1 28.5
								4 39.6 39.2 38.8 38.0
50	9.2324	73	9.2389	74	0.7611	9.9936	10	5 49.5 49.0 48.5 47.5
10 0	9.2397		9.2463		0.7537	9.9934	0 80	6 59.4 58.8 58.2 57.0
								7 69.3 68.6 67.9 66.5
	log cos	d	log cot	dc	log tan	log sin	′ °	8 79.2 78.4 77.6 76.0
								9 89.1 88.2 87.3 85.5

	94	93	91	89	87	86	85	84	82	81	79	78	77	76	75	74	73
1	9.4	9.3	9.1	8.9	8.7	8.6	8.5	8.4	8.2	8.1	7.9	7.8	7.7	7.6	7.5	7.4	7.3
2	18.8	18.6	18.2	17.8	17.4	17.2	17.0	16.8	16.4	16.2	15.8	15.6	15.4	15.2	15.0	14.8	14.6
3	28.2	27.9	27.3	26.7	26.1	25.8	25.5	25.2	24.6	2.34	23.7	23.4	23.1	22.8	22.5	22.2	21.9
4	37.6	37.2	36.4	35.6	34.8	34.4	34.0	33.6	32.8	32.4	31.6	31.2	30.8	30.4	30.0	29.6	29.2
5	47.0	46.5	45.5	44.5	43.5	43.0	42.5	42.0	41.0	40.5	39.5	39.0	38.5	38.0	37.5	37.0	36.5
6	56.4	55.8	54.6	53.4	52.2	51.6	51.0	50.4	49.2	48.6	47.4	46.8	46.2	45.6	45.0	44.4	43.8
7	65.8	65.1	63.7	62.3	60.9	60.2	59.5	58.8	57.4	56.7	55.3	54.6	53.9	53.2	52.5	51.8	51.1
8	75.2	74.4	72.8	71.2	69.6	68.8	68.0	67.2	65.6	64.8	63.2	62.4	61.6	60.8	60.0	59.2	58.4
9	84.6	83.7	81.9	80.1	78.3	77.4	76.5	75.6	73.8	72.9	71.1	70.2	69.3	68.4	67.5	66.6	65.7

Formulas for using Table inversely

$$\text{sin, tan, cot: } \begin{cases} \log x' = \log \sin x - S \\ \log x' = \log \tan x - T \\ \operatorname{colog} x' = \log \cot x - \text{co T} \end{cases} \qquad \text{cos: } \begin{cases} \log (90 - x)' = \log \cos x - S \\ \log (90 - x)' = \log \cot x - T \\ \operatorname{colog} (90 - x)' = \log \tan x - \text{co T} \end{cases}$$

Logarithms of Trigonometric Functions

° ′	log sin	d	log tan	dc	log cot	log cos	d		pp
10 0	9 2397	71	9.2463	73	0.7537	9.9934	3	0 80	
10	9.2468	70	9.2536	73	0.7464	9.9931	2	50	
20	9.2538	68	9.2609	71	0.7391	9.9929	2	40	
30	9.2606	68	9.2680	70	0.7320	9.9927	3	30	
40	9.2674	66	9.2750	69	0.7250	9.9924	2	20	
50	9.2740	66	9.2819	68	0.7181	9.9922	3	10	
11 0	9.2806	64	9.2887	66	0.7113	9.9919	2	0 79	
10	9.2870	64	9.2953	67	0.7047	9.9917	3	50	
20	9.2934	63	9.3020	65	0.6980	9.9914	2	40	
30	9.2997	61	9.3085	64	0.6915	9.9912	3	30	
40	9.3058	61	9.3149	63	0.6851	9.9909	2	20	
50	9.3119	60	9.3212	63	0.6788	9.9907	3	10	
12 0	9.3179	59	9.3275	61	0.6725	9.9904	3	0 78	
10	9.3238	58	9.3336	61	0.6664	9.9901	2	50	
20	9.3296	57	9.3397	61	0.6603	9.9899	3	40	
30	9.3353	57	9.3458	59	0.6542	9.9896	3	30	
40	9.3410	56	9.3517	59	0.6483	9.9893	3	20	
50	9.3466	55	9.3576	58	0.6424	9.9890	3	10	
13 0	9.3521	54	9.3634	57	0.6366	9.9887	3	0 77	
10	9.3575	54	9.3691	57	0.6309	9.9884	3	50	
20	9.3629	53	9.3748	56	0.6252	9.9881	3	40	
30	9.3682	52	9.3804	55	0.6196	9.9878	3	30	
40	9.3734	52	9.3859	55	0.6141	9.9875	3	20	
50	9.3786	51	9.3914	54	0.6086	9.9872	3	10	
14 0	9.3837	50	9.3968	53	0.6032	9.9869	3	0 76	
10	9.3887	50	9.4021	53	0.5979	9.9866	3	50	
20	9.3937	49	9.4074	53	0.5926	9.9863	4	40	
30	9.3986	49	9.4127	51	0.5873	9.9859	3	30	
40	9.4035	48	9.4178	52	0.5822	9.9856	3	20	
50	9.4083	47	9.4230	51	0.5770	9.9853	4	10	
15 0	9.4130		9.4281		0.5719	9.9849		0 75	
	log cos	d	log cot	dc	log tan	log sin	d	′ °	

pp															
	73	71													
1	7.3	7.1													
2	14.6	14.2													
3	21.9	21.3													
4	29.2	28.4													
5	36.5	35.5													
6	43.8	42.6													
7	51.1	49.7													
8	58.4	56.8													
9	65.7	63.9													
	70	69													
1	7.0	6.9													
2	14.0	13.8													
3	21.0	20.7													
4	28.0	27.6													
5	35.0	34.5													
6	42.0	41.4													
7	49.0	48.3													
8	56.0	55.2													
9	63.0	62.1													
	68	67													
1	6.8	6.7													
2	13.6	13.4													
3	20.4	20.1													
4	27.2	26.8													
5	34.0	33.5													
6	40.8	40.2													
7	47.6	46.9													
8	54.4	53.6													
9	61.2	60.3													
	66	65													
1	6.6	6.5													
2	13.2	13.0													
3	19.8	19.5													
4	26.4	26.0													
5	33.0	32.5													
6	39.6	39.0													
7	46.2	45.5													
8	52.8	52.0													
9	59.4	58.5													

	64	63	61	60	59	58	57	56	55	54	53	52	51	50	48	47
1	6.4	6.3	6.1	6.0	5.9	5.8	5.7	5.6	5.5	5.4	5.3	5.2	5.1	5.0	4.8	4.7
2	12.8	12.6	12.2	12.0	11.8	11.6	11.4	11.2	11.0	10.8	10.6	10.4	10.2	10.0	9.6	9.4
3	19.2	18.9	18.3	18.0	17.7	17.4	17.1	16.8	16.5	16.2	15.9	15.6	15.3	15.0	14.4	14.1
4	25.6	25.2	24.4	24.0	23.6	23.2	22.8	22.4	22.0	21.6	21.2	20.8	20.4	20.0	19.2	18.8
5	32.0	31.5	30.5	30.0	29.5	29.0	28.5	28.0	27.5	27.0	26.5	26.0	25.5	25.0	24.0	23.5
6	38.4	37.8	36.6	36.0	35.4	34.8	34.2	33.6	33.0	32.4	31.8	31.2	30.6	30.0	28.8	28.2
7	44.8	44.1	42.7	42.0	41.3	40.6	39.9	39.2	38.5	37.8	37.1	36.4	35.7	35.0	33.6	32.9
8	51.2	50.4	48.8	48.0	47.2	46.4	45.6	44.8	44.0	43.2	42.4	41.6	40.8	40.0	38.4	37.6
9	57.6	56.7	54.9	54.0	53.1	52.2	51.3	50.4	49.5	48.6	47.7	46.8	45.9	45.0	43.2	42.3

Logarithms of Trigonometric Functions

°	′	log sin	d	log tan	dc	log cot	log cos	d		pp.
15	0	9.4130		9.4281		0.5719	9.9849	3	0 75	
	10	9.4177	47	9.4331	50	0.5669	9.9846	3	50	
			46		50					50 49
	20	9.4223		9.4381		0.5619	9.9843	4	40	1 5.0 4.9
	30	9.4269	46	9.4430	49	0.5570	9.9839	3	30	2 10.0 9.8
	40	9.4314	45	9.4479	49	0.5521	9.9836	4	20	3 15.0 14.7
	50	9.4359	45		48					
16	0	9.4403	44	9.4527		0.5473	9.9832	4	10	4 20.0 19.6
	10	9.4447	44	9.4575	48	0.5425	9.9828	3	0 74	5 25.0 24.5
			44	9.4622	47	0.5378	9.9825	4	50	6 30.0 29.4
	20	9.4491			47					
	30	9.4533	42	9.4669		0.5331	9.9821	4	40	7 35.0 34.3
	40	9.4576	43	9.4716	47	0.5284	9.9817	3	30	8 40.0 39.2
			42	9.4762	46	0.5238	9.9814	4	20	9 45.0 44.1
	50	9.4618			46					
17	0	9.4659	41	9.4808		0.5192	9.9810	4	10	
	10	9.4700	41	9.4853	45	0.5147	9.9806	4	0 73	
			41	9.4898	45	0.5102	9.9802	4	50	
	20	9.4741			45					48 47
	30	9.4781	40	9.4943		0.5057	9.9798	4	40	1 4.8 4.7
	40	9.4821	40	9.4987	44	0.5013	9.9794	4	30	2 9.6 9.4
			40	9.5031	44	0.4969	9.9790	4	20	3 14.4 14.1
	50	9.4861			44			4		
18	0	9.4900	39	9.5075		0.4925	9.9786	4	10	4 19.2 18.8
	10	9.4939	39	9.5118	43	0.4882	9.9782	4	0 72	5 24.0 23.5
			38	9.5161	43	0.4839	9.9778	4	50	6 28.8 28.2
	20	9.4977			42					
	30	9.5015	38	9.5203		0.4797	9.9774	4	40	7 33.6 32.9
	40	9.5052	37	9.5245	42	0.4755	9.9770	5	30	8 38.4 37.6
			38	9.5287	42	0.4713	9.9765	4	20	9 43.2 42.3
	50	9.5090			42					
19	0	9.5126	36	9.5329		0.4671	9.9761	4	10	
	10	9.5163	37	9.5370	41	0.4630	9 9757	5	0	
			36	9.5411	41	0.4589	9.9752	4	50 71	46 45
	20	9.5199			40					1 4.6 4.5
	30	9.5235	36	9.5451		0.4549	9.9748	5	40	2 9.2 9.0
	40	9.5270	35	9.5491	40	0.4509	9 9743	4	30	3 13.8 13.5
			36	9.5531	40	0.4469	9.9739	5	20	
	50	9.5306			40					4 18.4 18.0
20	0	9.5341	35	9.5571		0.4429	9.9734	4	10	5 23.0 22.5
				9.5611	40	0.4389	9.9730		0 70	6 27.6 27.0
		log cos	d	log cot	dc	log tan	log sin	d	′ °	7 32.2 31.5
										8 36.8 36.0
										9 41.4 40.5

44	43	42
1 4.4	4.3	4.2
2 8.8	8.6	8.4
3 13.2	12.9	12.6
4 17.6	17.2	16.8
5 22.0	21.5	21.0
6 26.4	25.8	25.2
7 30.8	30.1	29.4
8 35.2	34.4	33.6
9 39.6	38.7	37.8

41	40	39	38
1 4.1	4.0	3.9	3.8
2 8.2	8.0	7.8	7.6
3 12.3	12.0	11.7	11.4
4 16.4	16.0	15.6	15.2
5 20.5	20.0	19.5	19.0
6 24.6	24.0	23.4	22.8
7 28.7	28.0	27.3	26.6
8 32.8	32.0	31.2	30.4
9 36.9	36.0	35.1	34.2

37	36	35
1 3.7	3.6	3.5
2 7.4	7.2	7.0
3 11.1	10.8	10.5
4 14.8	14.4	14.0
5 18.5	18.0	17.5
6 22.2	21.6	21.0
7 25.9	25.2	24.5
8 29.6	28.8	28.0
9 33.3	32.4	31.5

Logarithms of Trigonometric Functions

° ′	log sin	d	log tan	dc	log cot	log cos	d		pp
20 0	9.5341	34	9.5611	39	0.4389	9.9730	5	0 70	
10	9.5375	34	9.5650	39	0.4350	9.9725	4	50	
									4
20	9.5409	34	9.5689	38	0.4311	9.9721	5	40	1 0.4
30	9.5443	34	0.5727	39	0.4273	9.9716	5	30	2 0.8
40	9.5477	33	9.5766	38	0.4234	9.9711	5	20	3 1.2
50	9.5510	33	9.5804	38	0.4196	9.9706	4	10	4 1.6
21 0	9.5543	33	9.5842	37	0.4158	9.9702	5	0 69	5 2.0
10	9.5576	33	9.5879	38	0.4121	9.9697	5	50	6 2.4
20	9.5609	32	9.5917	37	0.4083	9.9692	5	40	7 2.8
30	9.5641	32	9.5954	37	0.4046	9.9687	5	30	8 3.2
40	9.5673	31	9.5991	37	0.4009	9.9682	5	20	9 3.6
50	9.5704	32	9.6028	36	0.3972	9.9677	5	10	
22 0	9.5736	31	9.6064	36	0.3936	9.9672	5	0 68	
10	9.5767	31	9.6100	36	0.3900	9.9667	6	50	
									5
20	9.5798	30	9.6136	36	0.3864	9.9661	5	40	1 0.5
30	9.5828	31	9.6172	36	0.3828	9.9656	5	30	2 1.0
40	9.5859	30	9.6208	35	0.3792	9.9651	5	20	3 1.5
50	9.5889	30	9.6243	36	0.3757	9.9646	6	10	4 2.0
23 0	9.5919	29	9.6279	35	0.3721	9.9640	5	0 67	5 2.5
10	9.5948	30	9.6314	34	0.3686	9.9635	6	50	6 3.0
20	9.5978	29	9.6348	35	0.3652	9.9629	5	40	7 3.5
30	9.6007	29	9.6383	34	0.3617	9.9624	6	30	8 4.0
40	9.6036	29	9.6417	35	0.3583	9.9618	5	20	9 4.5
50	9.6065	28	9.6452	34	0.3548	9.9613	6	10	
24 0	9.6093	28	9.6486	34	0.3514	9.9607	5	0 66	
10	9.6121	28	9.6520	33	0.3480	9.9602	6	50	6
									1 0.6
20	9.6149	28	9.6553	34	0.3447	9.9596	6	40	2 1.2
30	9.6177	28	9.6587	33	0.3413	9.9590	6	30	3 1.8
40	9.6205	27	9.6620	34	0.3380	9.9584	5	20	
									4 2.4
50	9.6232	27	9.6654	33	0.3346	9.9579	6	10	5 3.0
25 0	9.6259		9.6687		0.3313	9.9573		0 65	6 3.6
									7 4.2
	log cos	d	log cot	dc	log tan	log sin	d	′ °	8 4.8
									9 5.4

	39	38	37	36	35	34	33	32	31	30	29	28	27
1	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1	3.0	2.9	2.8	2.7
2	7.8	7.6	7.4	7.2	7.0	6.8	6.6	6.4	6.2	6.0	5.8	5.6	5.4
3	11.7	11.4	11.1	10.8	10.5	10.2	9.9	9.6	9.3	9.0	8.7	8.4	8.1
4	15.6	15.2	14.8	14.4	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.2	10.8
5	19.5	19.0	18.5	18.0	17.5	17.0	16.5	16.0	15.5	15.0	14.5	14.0	13.5
6	23.4	22.8	22.2	21.6	21.0	20.4	19.8	19.2	18.6	18.0	17.4	16.8	16.2
7	27.3	26.6	25.9	25.2	24.5	23.8	23.1	22.4	21.7	21.0	20.3	19.6	18.9
8	31.2	30.4	29.6	28.8	28.0	27.2	26.4	25.6	24.8	24.0	23.2	22.4	21.6
9	35.1	34.2	33.3	32.4	31.5	30.6	29.7	28.8	27.9	27.0	26.1	25.2	24.3

Logarithms of Trigonometric Functions

° ′	log sin	d	log tan	dc	log cot	log cos	d		pp
25 0	9.6259	27	9.6687	33	0.3313	9.9573	6	0 65	
10	9.6286	27	9.6720	32	0.3280	9.9567	6	50	
20	9.6313	27	9.6752	33	0.3248	9.9561	6	40	
30	9.6340	26	9.6785	32	0.3215	9.9555	6	30	6
40	9.6366	26	9.6817	33	0.3183	9.9549	6	20	1 0.6
50	9.6392	26	9.6850	32	0.3150	9.9543	6	10	2 1.2
26 0	9.6418	26	9.6882	32	0.3118	9.9537	7	0 64	3 1.8
10	9.6444	26	9.6914	32	0.3086	9.9530	6	50	4 2.4
20	9.6470	25	9.6946	31	0.3054	9.9524	6	40	5 3.0
30	9.6495	26	9.6977	32	0.3023	9.9518	6	30	6 3.6
40	9.6521	25	9.7009	31	0.2991	9.9512	7	20	7 4.2
50	9.6546	24	9.7040	32	0.2960	9.9505	6	10	8 4.8
27 0	9.6570	25	9.7072	31	0.2928	9.9499	7	0 63	9 5.4
10	9.6595	25	9.7103	31	0.2897	9.9492	6	50	
20	9.6620	24	9.7134	31	0.2866	9.9486	7	40	7
30	9.6644	24	9.7165	31	0.2835	9.9479	6	30	1 0.7
40	9.6668	24	9.7196	30	0.2804	9.9473	7	20	2 1.4
50	9.6692	24	9.7226	31	0.2774	9.9466	7	10	3 2.1
28 0	9.6716	24	9.7257	30	0.2743	9.9459	6	0 62	4 2.8
10	9.6740	23	9.7287	30	0.2713	9.9453	7	50	5 3.5
20	9.6763	24	9.7317	31	0.2683	9.9446	7	40	6 4.2
30	9.6787	23	9.7348	30	0.2652	9.9439	7	30	7 4.9
40	9.6810	23	9.7378	30	0.2622	9.9432	7	20	8 5.6
50	9.6833	23	9.7408	30	0.2592	9.9425	7	10	9 6.3
29 0	9.6856	22	9.7438	29	0.2562	9.9418	7	0 61	
10	9.6878	23	9.7467	30	0.2533	9.9411	7	50	8
20	9.6901	22	9.7497	29	0.2503	9.9404	7	40	1 0.8
30	9.6923	23	9.7526	30	0.2474	9.9397	7	30	2 1.6
40	9.6946	22	9.7556	29	0.2444	9.9390	7	20	3 2.4
50	9.6968	22	9.7585	29	0.2415	9.9383	8	10	4 3.2
30 0	9.6990		9.7614		0.2386	9.9375		0 60	5 4.0
									6 4.8
	log cos	d	log cot	dc	log tan	log sin	d	′ °	7 5.6
									8 6.4
									9 7.2

	33	32	31
1	3.3	3.2	3.1
2	6.6	6.4	6.2
3	9.9	9.6	9.3
4	13.2	12.8	12.4
5	16.5	16.0	15.5
6	19.8	19.2	18.6
7	23.1	22.4	21.7
8	26.4	25.6	24.8
9	29.7	28.8	27.9

	30	29	27	26	25
1	3.0	2.9	2.7	2.6	2.5
2	6.0	5.8	5.4	5.2	5.0
3	9.0	8.7	8.1	7.8	7.5
4	12.0	11.6	10.8	10.4	10.0
5	15.0	14.5	13.5	13.0	12.5
6	18.0	17.4	16.2	15.6	15.0
7	21.0	20.3	18.9	18.2	17.5
8	24.0	23.2	21.6	20.8	20.0
9	27.0	26.1	24.3	23.4	22.5

	24	23	22
1	2.4	2.3	2.2
2	4.8	4.6	4.4
3	7.2	6.9	6.6
4	9.6	9.2	8.8
5	12.0	11.5	11.0
6	14.4	13.8	13.2
7	16.8	16.1	15.4
8	19.2	18.4	17.6
9	21.6	20.7	19.8

Logarithms of Trigonometric Functions

° ′	log sin	d	log tan	dc	log cot	log cos	d		pp
30 0	9.6990		9.7614		0.2386	9.9375	7	0 60	
10	9.7012	22	9.7644	30	0.2356	9.9368	7	50	
20	9.7033	22	9.7673	28	0.2327	9.9361	8	40	7
30	9.7055	21	9.7701	29	0.2299	9.9353	7	30	1 0.7
40	9.7076	21	9.7730	29	0.2270	9.9346	8	20	2 1.4
									3 2.1
50	9.7097	21	9.7759	29	0.2241	9.9338	7	10	
31 0	9.7118	21	9.7788	28	0.2212	9.9331	8	0 59	4 2.8
10	9.7139	21	9.7816	29	0.2184	9.9323	8	50	5 3.5
									6 4.2
20	9.7160	21	9.7845	28	0.2155	9.9315	7	40	
30	9.7181	20	9.7873	29	0.2127	9.9308	8	30	7 4.9
40	9.7201	21	9.7902	28	0.2098	9.9300	8	20	8 5.6
									9 6.3
50	9.7222	20	9.7930	28	0.2070	9.9292	8	10	
32 0	9.7242	20	9.7958	28	0.2042	9.9284	8	0 58	
10	9.7262	20	9.7986	28	0.2014	9.9276	8	50	8
									1 0.8
20	9.7282	20	9.8014	28	0.1986	9.9268	8	40	2 1.6
30	9.7302	20	9.8042	28	0.1958	9.9260	8	30	3 2.4
40	9.7322	20	9.8070	27	0.1930	9.9252	8	20	
									4 3.2
50	9.7342	19	9.8097	28	0.1903	9.9244	8	10	5 4.0
33 0	9.7361	19	9.8125	28	0.1875	9.9236	8	0 57	6 4.8
10	9.7380	20	9.8153	27	0.1847	9.9228	9	50	
									7 5.6
20	9.7400	19	9.8180	28	0.1820	9.9219	8	40	8 6.4
30	9.7419	19	9.8208	27	0.1792	9.9211	8	30	9 7.2
40	9.7438	19	9.8235	28	0.1765	9.9203	9	20	
									9
50	9.7457	19	9.8263	27	0.1737	9.9194	8	10	1 0.9
34 0	9.7476	18	9.8290	27	0.1710	9.9186	9	0 56	2 1.8
10	9.7494	19	9.8317	27	0.1683	9.9177	8	50	3 2.7
20	9.7513	18	9.8344	27	0.1656	9.9169	9	40	4 3.6
30	9.7531	19	9.8371	27	0.1629	9.9160	9	30	5 4.5
40	9.7550	18	9.8398	27	0.1602	9.9151	9	20	6 5.4
50	9.7568	18	9.8425	27	0.1575	9.9142	8	10	7 6.3
35 0	9.7586		9.8452		0.1548	9.9134		0 55	8 7.2
									9 8.1
	log cos	d	log cot	dc	log tan	log sin	d	′ °	

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	22	21
1	2.7	2.2	2.1
2	5.4	4.4	4.2
3	8.1	6.6	6.3
4	10.8	8.8	8.4
5	13.5	11.0	10.5
6	16.2	13.2	12.6
7	18.9	15.4	14.7
8	21.6	17.6	16.8
9	24.3	19.8	18.9

	20	19	18
1	2.0	1.9	1.8
2	4.0	3.8	3.6
3	6.0	5.7	5.4
4	8.0	7.6	7.2
5	10.0	9.5	9.0
6	12.0	11.4	10.8
7	14.0	13.3	12.6
8	16.0	15.2	14.4
9	18.0	17.1	16.2

Logarithms of Trigonometric Functions

° ′	log sin	d	log tan	dc	log cot	log cos	d		pp
35 0	9.7586	18	9.8452	27	0.1548	9.9134	9	0 55	
10	9.7604	18	9.8479	27	0.1521	9.9125	9	50	
									9
20	9.7622	18	9.8506	27	0.1494	9.9116	9	40	1 0.9
30	9.7640	17	9.8533	26	0.1467	9.9107	9	30	2 1.8
40	9.7657	18	9.8559	27	0.1441	9.9098	9	20	3 2.7
50	9.7675	17	9.8586	27	0.1414	9.9089	9	10	4 3.6
36 0	9.7692	18	9.8613	26	0.1387	9.9080	10	0 54	5 4.5
10	9.7710	17	9.8639	27	0.1361	9.9070	9	50	6 5.4
20	9.7727	17	9.8666	26	0.1334	9.9061	9	40	7 6.3
30	9.7744	17	9.8692	26	0.1308	9.9052	10	30	8 7.2
40	9.7761	17	9.8718	27	0.1282	9.9042	9	20	9 8.1
50	9.7778	17	9.8745	26	0.1255	9.9033	10	10	
37 0	9.7795	16	9.8771	26	0.1229	9.9023	9	0 53	10
10	9.7811	17	9.8797	27	0.1203	9.9014	10	50	1 1.0
									2 2.0
20	9.7828	16	9.8824	26	0.1176	9.9004	9	40	3 3.0
30	9.7844	17	9.8850	26	0.1150	9.8995	10	30	
40	9.7861	16	9.8876	26	0.1124	9.8985	10	20	4 4.0
50	9.7877	16	9.8902	26	0.1098	9.8975	10	10	5 5.0
38 0	9.7893	17	9.8928	26	0.1072	9.8965	10	0 52	6 6.0
10	9.7910	16	9.8954	26	0.1046	9.8955	10	50	7 7.0
									8 8.0
20	9.7926	15	9.8980	26	0.1020	9.8945	10	40	9 9.0
30	9.7941	16	9.9006	26	0.0994	9.8935	10	30	
40	9.7957	16	9.9032	26	0.0968	9.8925	10	20	
50	9.7973	16	9.9058	26	0.0942	9.8915	10	10	11
39 0	9.7989	15	9.9084	26	0.0916	9.8905	10	0 51	1 1.1
10	9.8004	16	9.9110	25	0.0890	9.8895	11	50	2 2.2
									3 3.3
20	9.8020	15	9.9135	26	0.0865	9.8884	10	40	4 4.4
30	9.8035	15	9.9161	26	0.0839	9.8874	10	30	5 5.5
40	9.8050	16	9.9187	25	0.0813	9.8864	11	20	6 6.6
50	9.8066	15	9.9212	26	0.0788	9.8853	10	10	7 7.7
40 0	9.8081		9.9238		0.0762	9.8843		0 50	8 8.8
									9 9.9
	log cos	d	log cot	dc	log tan	log sin	d	′ °	

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25	18	17
1	2.5	1.8	1.7
2	5.0	3.6	3.4
3	7.5	5.4	5.1
4	10.0	7.2	6.8
5	12.5	9.0	8.5
6	15.0	10.8	10.2
7	17.5	12.6	11.9
8	20.0	14.4	13.6
9	22.5	16.2	15.3

	16	15
1	1.6	1.5
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5

Logarithms of Trigonometric Functions

°	′	log sin	d	log tan	dc	log cot	log cos	d			pp
40	0	9.8081		9.9238		0.0762	9.8843	11	0	50	
	10	9.8096	15	9.9264	26	0.0736	9.8832	11	50		
			15		25						
	20	9.8111		9.9289		0.0711	9.8821	11	40		10
	30	9.8125	14	9.9315	26	0.0685	9.8810	10	30		1 1.0
	40	9.8140	15	9.9341	26	0.0659	9.8800	11	20		2 2.0
			15		25						3 3.0
	50	9.8155		9.9366		0.0634	9.8789	11	10		
41	0	9.8169	14	9.9392	26	0.0608	9.8778	11	0	49	4 4.0
	10	9.8184	15	9.9417	25	0.0583	9.8767	11	50		5 5.0
			14		26						6 6.0
	20	9.8198		9.9443		0.0557	9.8756	11	40		7 7.0
	30	9.8213	15	9.9468	25	0.0532	9.8745	12	30		8 8.0
	40	9.8227	14	9.9494	26	0.0506	9.8733	11	20		9 9.0
			14		25						
	50	9.8241		9.9519		0.0481	9.8722	11	10		
42	0	9.8255	14	9.9544	25	0.0456	9.8711	12	0	48	
	10	9.8269	14	9.9570	26	0.0430	9.8699	11	50		
			14		25						11
	20	9.8283		9.9595		0.0405	9.8688	12	40		1 1.1
	30	9.8297	14	9.9621	26	0.0379	9.8676	11	30		2 2.2
	40	9.8311	14	9.9646	25	0.0354	9.8665	12	20		3 3.3
			13		25						4 4.4
	50	9.8324		9.9671		0.0329	9.8653	12	10		5 5.5
43	0	9.8338	14	9.9697	26	0.0303	9.8641	12	0	47	6 6.6
	10	9.8351	13	9.9722	25	0.0278	9.8629	11	50		
			14		25						7 7.7
	20	9.8365		9.9747		0.0253	9.8618	12	40		8 8.8
	30	9.8378	13	9.9772	25	0.0228	9.8606	12	30		9 9.9
	40	9.8391	13	9.9798	26	0.0202	9.8594	12	20		
			14		25						
	50	9.8405		9.9823		0.0177	9.8582	13	10		
44	0	9.8418	13	9.9848	25	0.0152	9.8569	12	0	46	
	10	9.8431	13	9.9874	26	0.0126	9.8557	12	50		12
			13		25						1 1.2
	20	9.8444		9.9899		0.0101	9.8545	13	40		2 2.4
	30	9.8457	13	9.9924	25	0.0076	9.8532	12	30		3 3.6
	40	9.8469	12	9.9949	25	0.0051	9.8520	13	20		4 4.8
			13		26						5 6.0
	50	9.8482		9.9975		0.0025	9.8507	12	10		6 7.2
45	0	9.8495	13	0.0000	25	0.0000	9.8495		0	45	7 8.4
											8 9.6
											9 10.8
		log cos	d	log cot	dc	log tan	log sin	d	′	°	

26		25		15		14		13	
1	2.6	1	2.5	1	1.5	1	1.4	1	1.3
2	5.2	2	5.0	2	3.0	2	2.8	2	2.6
3	7.8	3	7.5	3	4.5	3	4.2	3	3.9
4	10.4	4	10.0	4	6.0	4	5.6	4	5.2
5	13.0	5	12.5	5	7.5	5	7.0	5	6.5
6	15.6	6	15.0	6	9.0	6	8.4	6	7.8
7	18.2	7	17.5	7	10.5	7	9.8	7	9.1
8	20.8	8	20.0	8	12.0	8	11.2	8	10.4
9	23.4	9	22.5	9	13.5	9	12.6	9	11.7

Natural Trigonometric Functions

Deg.	Radians	n sin	n csc	n tan	n cot	n sec	n cos		
0	0.0000	.000	.....	.000	.....	1.000	1.00	1.5708	90
1	0.0175	.017	57.3	.017	57.3	1.000	1.00	1.5533	89
2	0.0349	.035	28.7	.035	28.6	1.001	.999	1.5359	88
3	0.0524	.052	19.1	.052	19.1	1.001	.999	1.5184	87
4	0.0698	.070	14.3	.070	14.3	1.002	.998	1.5010	86
5	0.0873	.087	11.5	.087	11.4	1.004	.996	1.4835	85
6	0.1047	.105	9.57	.105	9.51	1.006	.995	1.4661	84
7	0.1222	.122	8.21	.123	8.14	1.008	.993	1.4486	83
8	0.1396	.139	7.19	.141	7.12	1.010	.990	1.4312	82
9	0.1571	.156	6.39	.158	6.31	1.012	.988	1.4137	81
10	0.1745	.174	5.76	.176	5.67	1.015	.985	1.3963	80
11	0.1920	.191	5.24	.194	5.14	1.019	.982	1.3788	79
12	0.2094	.208	4.81	.213	4.70	1.022	.978	1.3614	78
13	0.2269	.225	4.45	.231	4.33	1.026	.974	1.3439	77
14	0.2443	.242	4.13	.249	4.01	1.031	.970	1.3265	76
15	0.2618	.259	3.86	.268	3.73	1.035	.966	1.3090	75
16	0.2793	.276	3.63	.287	3.49	1.040	.961	1.2915	74
17	0.2967	.292	3.42	.306	3.27	1.046	.956	1.2741	73
18	0.3142	.309	3.24	.325	3.08	1.051	.951	1.2566	72
19	0.3316	.326	3.07	.344	2.90	1.058	.946	1.2392	71
20	0.3491	.342	2.92	.364	2.75	1.064	.940	1.2217	70
21	0.3665	.358	2.79	.384	2.61	1.071	.934	1.2043	69
22	0.3840	.375	2.67	.404	2.48	1.079	.927	1.1868	68
23	0.4014	.391	2.56	.424	2.36	1.086	.921	1.1694	67
24	0.4189	.407	2.46	.445	2.25	1.095	.914	1.1519	66
25	0.4363	.423	2.37	.466	2.14	1.103	.906	1.1345	65
26	0.4538	.438	2.28	.488	2.05	1.113	.899	1.1170	64
27	0.4712	.454	2.20	.510	1.96	1.122	.891	1.0996	63
28	0.4887	.469	2.13	.532	1.88	1.133	.883	1.0821	62
29	0.5061	.485	2.06	.554	1.80	1.143	.875	1.0647	61
30	0.5236	.500	2.00	.577	1.73	1.155	.866	1.0472	60
31	0.5411	.515	1.94	.601	1.66	1.167	.857	1.0297	59
32	0.5585	.530	1.89	.625	1.60	1.179	.848	1.0123	58
33	0.5760	.545	1.84	.649	1.54	1.192	.839	0.9948	57
					1.50				30'
34	0.5934	.559	1.79	.675	1.48	1.206	.829	0.9774	56
35	0.6109	.574	1.74	.700	1.43	1.221	.819	0.9599	55
36	0.6283	.588	1.70	.727	1.38	1.236	.809	0.9425	54
37	0.6458	.602	1.66	.754	1.33	1.252	.799	0.9250	53
38	0.6632	.616	1.62	.781	1.28	1.269	.788	0.9076	52
39	0.6807	.629	1.59	.810	1.23	1.287	.777	0.8901	51
40	0.6981	.643	1.56	.839	1.19	1.305	.766	0.8727	50
41	0.7156	.656	1.52	.869	1.15	1.325	.755	0.8552	49
42	0.7330	.669	1.49	.900	1.11	1.346	.743	0.8378	48
43	0.7505	.682	1.47	.933	1.07	1.367	.731	0.8203	47
44	0.7679	.695	1.44	.966	1.04	1.390	.719	0.8029	46
45	0.7854	.707	1.41	1.00	1.00	1.414	.707	0.7854	45
		n cos	n sec	n cot	n tan	n csc	n sin	Radians	Deg.

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3305	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

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rational numbe $\frac{x}{a}$

---

polynomial

remainde -

$$\frac{f(x)}{x-a} = \frac{Q(x)}{} + \frac{f(a)}{x-a}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$p = \frac{}{1 + e\cos\theta}$$

degrees.

	Sin	cos	tan	cot	sec	csc	
0°							
90°							
180°							

to solve

$$x^3 - \frac{13}{3}x + \frac{97}{27} = 0$$

$$y = x^3$$
$$y = \frac{13}{3}x - \frac{97}{27}$$ } graph

Simple Harmonic

$T = \frac{2\pi}{K}$ $\quad x = a\cos(Kt - \epsilon)$
$y = a\sin(Kt - \epsilon)$
Moving waves

$N =$

$h = \frac{2\pi}{L}$

$K = \frac{2\pi}{T}$

$y = a\sin(hx - Kt)$

Standing Wave

Cardan Solution

Tartaglia

$$x^3 + ax + b = 0$$

$$x^3 = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}}$$

Abel in 1821 5° or higher by other than algebra

To eliminate x^2

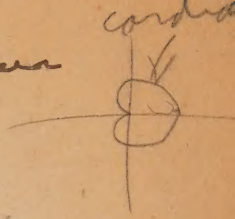
$y = x - k$ - substitute in origin

$k = -\frac{a}{3}$ (coef. of x^2)

rotation page 361

$\log e = .4343$

for note book.



polar and cartesian coordinates

$\rho = \sqrt{x^2+y^2}$

$\theta = \arctan \frac{y}{x}$

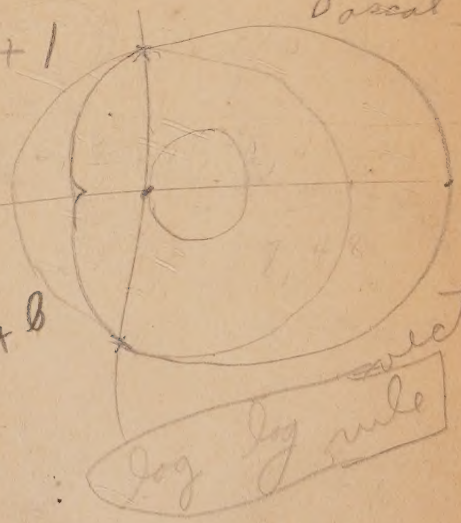
Limaçon
Pascal

$\rho = \cos\theta + 1$

$\rho = \cos\frac{\theta}{2}$

$\rho = \sin\frac{\theta}{2}$

$\rho = a\left\{\begin{matrix}\sin\theta\\ \cos\theta\end{matrix}\right\} + b$



log log rule

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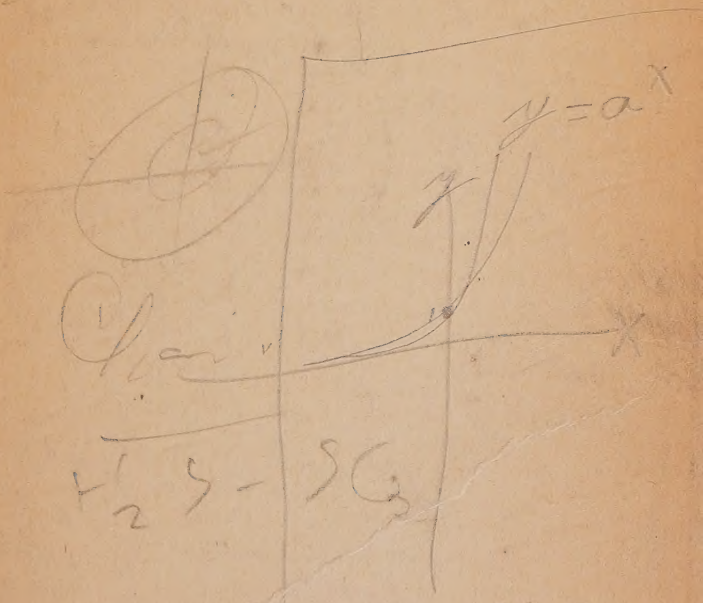
Ralph Schmiedlin
2810 Oakridge Ave.
Madison, Wis.

θ = parameter

Parametric Eq. of $\odot$



e = val of a which makes a slope of $a^x = 1$ at (0

e = 2.718